BOUNDED GAPS BETWEEN PRIMES

Searching for small gaps between consecutive primes is one way to approach the twin primes conjecture, one of the most celebrated unsolved problems in number theory. This book documents the remarkable developments of recent decades, whereby an upper bound on the known gap length between infinite numbers of consecutive primes has been reduced to a tractable finite size. The text is both introductory and comprehensive: the detailed way in which results are proved is fully set out and plenty of background material is included. The reader journeys from selected historical theorems to the latest best result, exploring the contributions of a vast array of mathematicians, including Bombieri, Goldston, Motohashi, Pintz, Yildirim, Zhang, Maynard, Tao and Polymath8. The book is supported by a linked and freely available package of computer programs. The material is suitable for graduate students and of interest to any mathematician curious about recent breakthroughs in the field.

KEVIN BROUGHAN is Emeritus Professor at the University of Waikato, New Zealand. He cofounded and is a fellow of the New Zealand Mathematical Society. Broughan brings a unique set of knowledge and skills to this project, including number theory, analysis, topology, dynamical systems and computational mathematics. He previously authored the two-volume work *Equivalents of the Riemann Hypothesis* (Cambridge University Press, 2017) and wrote a software package which is part of Goldfeld's *Automorphic Forms and L-Functions or the Group* GL(n,R) (Cambridge University Press, 2006).

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The Epic Breakthroughs of the Early Twenty-First Century

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Dedicated to Jackie, Jude and Beck

Pintz took a closer look at the flawed proof and came up with a key insight for the ultimate fix. He contacted Goldston in December 2004, and the three number theorists, Goldston, Pintz and Yildirim, had a complete proof by early February. They circulated the manuscript to a handful of experts . . . One of these, Motohashi, found a shortcut that led to a surprising short proof of the basic qualitative result.

– Science Magazine

In April 2013, a lecturer at the University of New Hampshire submitted a paper to the Annals of Mathematics. Within weeks word spread – a little-known mathematician, with no permanent job. The rest is history.

– IMDb

In November 2013, inspired by Zhang's extraordinary breakthrough, James Maynard dramatically slashed the bound (for an infinite number of prime pairs) to 600*, by a substantially easier method. Both Maynard and Terry Tao, who had independently developed the same idea, were able to extend their proofs to show that for any given integer* $m \geq 1$ *there exists a bound such that there are infinitely many intervals of that length containing at least* m *distinct primes. If Zhang's method is combined with the Maynard–Tao setup, then it appears that the bound can be further reduced to* 246*. If all of these techniques could be pushed to their limit, then we would obtain a bound of* 12 *(or arguably* 6*), so new ideas are still needed to have a feasible plan for proving the twin prime conjecture.*

– Andrew Granville

The conquest of the bounded gaps is an historical event that will continue to attract attention long into the future.

– Yoichi Motohashi

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Preface

This book has been written to mark, celebrate and detail the remarkable developments which occurred in the number theory area of gaps between primes in the first two decades of the twenty-first century and were published in the main between 2006 and 2015. Many mathematicians contributed to these developments, and most of the main participants are acknowledged and their work set out in Chapters 4 through 8.

In addition, there is an introductory chapter giving background material including an overview of the developments, their historical context, technical prerequisites and a brief guide to introductory material. The second chapter is also introductory, with details of the Brun and Selberg sieves, and a third gives details of a selection of early work relating to prime gaps.

The main part of the book, Chapters 4 through 7, is time sequential, reporting successively the work of Goldston, Motohashi, Pintz and Yildirim, then Motohashi, Pintz and Zhang, then Maynard, and finally Polymath8b. Each chapter is roughly self-contained, even if ideas from one advance sparked discoveries in the next, and the reader might wish to dwell especially on Chapter 6 (Maynard) and/or its "completion" Chapter 7 (Polymath8b), where the prime gap bound 246 is derived.

Chapter 8 is devoted in part to an exposition of Zhang's variation, but mostly to Polymath8a's variations of the Bombieri–Vinogradov theorem. An extensive set of appendices support the work of that chapter, and even this is insufficient to fully report on the work, which would take a similar sized book to this. The final chapter takes a different course than the others, giving a summary of further developments which have resulted already from the developments in the short space of time since 2015.

To aid the reader, definitions are repeated in various places, and major steps in proofs numbered to give a clear indication of the main parts and allow for easy proof internal referencing. For the most part, we have kept to the methods and logic of the original authors, while clarifying the argument when possible and providing background results when this seemed useful for readers.

The reader should be warned this is not an introductory number theory text. The expected background for a beginner would be a study of elementary analytic number theory, especially that relating to the distribution of the primes, and complex calculus, especially for Chapter 4. The Fourier transform is used intensively in Chapter 7. There are excellent texts from which a selection might be made. For example, those by Apostol [4], Hardy and Wright [90], Nathanson [152, 153], De Koninck and Luca [121] and especially Montgomery and Vaughan [144].

We use the arithmetic functions $\mu(n)$,$\varphi(n)$, $\Omega(n)$, $\omega(n)$ and the Landau–Vinogradov expressions

$$f(x) \ll g(x),\ f(x) = O(g(x)),\ f(x) = o(g(x)),$$

each with its usual meaning.

In writing a book of this nature, it is somewhat tempting to dwell on particular results and see if they could be improved. Too much of that ensures the work would never be finished and has been resisted. However, there are some small improvements. For example, proofs of some lemmas have been provided, such as Lemma 2.21, the sum of log p/p over the divisors of an integer n in Chapter 2; the derivation of a particularly nasty binomial sum identity in Lemma 4.3 in Chapter 4; a clearer presentation of the derivation of Zhang's lower bound given comments in his introduction which are apparently misleading, showing that Maynard's choices of parameters in two of his explicit forms is optimal for Theorem 6.17 and Lemma 6.11; and the derivation of a bound for smaller values of ϵ to extend Theorem 7.17.

Many computations have been repeated, so numerical results may at times look a little different from published results. There is a website for errata and notes, and readers are encouraged to communicate with the author in this regard at `kab@waikato.ac.nz`. The website is linked to:

www.math.waikato.ac.nz/∼kab

Also linked to this website is the suite of Mathematica™ programs, PGpack, related to the material in this volume, which is available for free download. Instructions on how to download the software are given in Appendix I. The functions `RayleighQuotientMaynard` and `RayleighQuotientPolymath` are essentially those of Maynard and Polymath respectively. The application of these functions is an essential part of the derivation of their results. Readers should be aware that the result along the lines of that of Maynard was shown by Polymath to be optimal, and the author was not able to improve the result of the Polymath, even though, on the face of it, this should be possible. Consulting the epilogue in Chapter 9 could be useful in this regard.

Even though I have done my best to properly acknowledge the contributions of those whose results have made this work possible, a range of limitations mean there will undoubtedly be omissions, for which I am regretful. One of the functions of the book web page "Errata and Notes" is to provide a remedy in such situations, and readers are encouraged to contact the author at *kab@waikato.ac.nz* if necessary.

It is not the purpose of this book to detail the story of bounded gaps between primes as a human endeavour, just the technical details and then only part of these. The human side, the "backstory", is of great interest, and, according to this author, quite unique in the history of mathematics, and even more generally of science. It should be told before too many years have elapsed and memory blurs the edges of details. It already has begun to do that – it is a natural process.

The human aspect of the prime gaps discoveries has many components – group and individual creativity, generous leadership, courage in the face of error, determination to solve problems over decades, rivalry and disappointment, deserved and undeserved fame, excellent and poor communication, generosity in giving credit and obstinacy when it comes to attribution. There is a link established on the book home page, entitled Backstory, giving some summary information and further links. I have started collecting these and they should grow in number: for example, articles in *Science* and *Quanta* magazines, the Notices and Bulletin of the American Mathematical Society (AMS), in the Newsletter of the European Math Union (EMU), and in *Ideas of the IAS* (Institute of Advanced Studies, Princeton). In addition, links to a Csicsery film, to the book by Vicky Neal, and to Polymath8's home page with of the order of thousand further links. Finally, this web-based information would not in any way take the place of a coherent write-up, preparation for which would involve extensive interviewing as well as investigating primary and secondary sources.

Kevin Broughan

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I need to emphasize that the greatest contribution to this work belongs to those responsible for the remarkable breakthroughs which have been reported: in the main (in chronological order) to Bombieri, Davenport, Goldston, Pintz, Yildirim, Motohashi, Zhang, Polymath8a, Maynard, Tao and Polymath8b. Note that in Section 1.4 there is a list of some of the principal contributors to the two Polymath8 projects.

Associated software PGpack is freely available from the author's website under the GNU general public license. It was written by the author with two exceptions which are kindly acknowledged. The functions RayleighQuotientMaynard and Rayleigh QuotientPolymath are adaptations of the software written and used by James Maynard and Polymath8b respectively.

Cover image: Doubtful Sound is a fjord on the south coast of the South Island of New Zealand. It was named by the explorer James Cook in 1770 as Doubtful Harbour. (Photo: Terry Latson / 500px / Getty Images.)