CAMBRIDGE STUDIES IN ADVANCED MATHEMATICS 212

ANALYTIC COMBINATORICS IN SEVERAL VARIABLES

Discrete structures model a vast array of objects ranging from DNA sequences to internet networks. The theory of generating functions provides an algebraic framework for discrete structures to be enumerated using mathematical tools. This book is the result of 25 years of work developing analytic machinery to recover asymptotics of multivariate sequences from their generating functions, using multivariate methods that rely on a combination of analytic, algebraic, and topological tools. The resulting theory of analytic combinatorics in several variables is put to use in diverse applications from mathematics, combinatorics, computer science, and the natural sciences. This new edition is even more accessible to graduate students, with many more exercises, computational examples with Sage worksheets to illustrate the main results, updated background material, additional illustrations, and a new chapter providing a conceptual overview.

Robin Pemantle is Merriam Term Professor of Mathematics at the University of Pennsylvania, working in the fields of probability theory and combinatorics. He received his bachelor's degree from Berkeley and his PhD from MIT. He is a Fellow of the AMS and IMS and a winner of the Rollo Davidson Prize.

Mark C. Wilson is Senior Teaching Faculty at the College of Information and Computer Sciences at the University of Massachusetts, Amherst. He received his PhD in mathematics from the University of Wisconsin–Madison. He is Editor-in-Chief of *Notices of the American Mathematical Society* and life member of the Combinatorial Mathematics Society of Australasia.

Stephen Melczer is Assistant Professor in the Department of Combinatorics and Optimization at the University of Waterloo. He received doctorates from the École normale supérieure de Lyon and the University of Waterloo. He is a recipient of a Governor General Silver Academic Medal and previously published the textbook *An Invitation to Analytic Combinatorics*.

CAMBRIDGE STUDIES IN ADVANCED MATHEMATICS

All the titles listed below can be obtained from good booksellers or from Cambridge University Press. For a complete series listing, visit www.cambridge.org/mathematics.

Already Published

172. T. Ceccherini-Silberstein, F. Scarabotti & F. Tolli *Discrete Harmonic Analysis*
173. Garrett *Modern Analysis of Automorphic Forms by Example, I*
174. Garrett *Modern Analysis of Automorphic Forms by Example, II*
175. G. Navarro *Character Theory and the McKay Conjecture*
176. P. Fleig, H. P. A. Gustafsson, A. Kleinschmidt & D. Persson *Eisenstein Series and Automorphic Representations*
177. E. Peterson *Formal Geometry and Bordism Operators*
178. A. Ogus *Lectures on Logarithmic Algebraic Geometry*
179. N. Nikolski *Hardy Spaces*
180. D.-C. Cisinski *Higher Categories and Homotopical Algebra*
181. A. Agrachev, D. Barilari & U. Boscain *A Comprehensive Introduction to Sub-Riemannian Geometry*
182. N. Nikolski *Toeplitz Matrices and Operators*
183. A. Yekutieli *Derived Categories*
184. C. Demeter *Fourier Restriction, Decoupling and Applications*
185. D. Barnes & C. Roitzheim *Foundations of Stable Homotopy Theory*
186. V. Vasyunin & A. Volberg *The Bellman Function Technique in Harmonic Analysis*
187. M. Geck & G. Malle *The Character Theory of Finite Groups of Lie Type*
188. B. Richter *Category Theory for Homotopy Theory*
189. R. Willett & G. Yu *Higher Index Theory*
190. A. Bobrowski *Generators of Markov Chains*
191. D. Cao, S. Peng & S. Yan *Singularly Perturbed Methods for Nonlinear Elliptic Problems*
192. E. Kowalski *An Introduction to Probabilistic Number Theory*
193. V. Gorin *Lectures on Random Lozenge Tilings*
194. E. Riehl & D. Verity *Elements of ∞-Category Theory*
195. H. Krause *Homological Theory of Representations*
196. F. Durand & D. Perrin *Dimension Groups and Dynamical Systems*
197. A. Sheffer *Polynomial Methods and Incidence Theory*
198. T. Dobson, A. Malnič & D. Marušič *Symmetry in Graphs*
199. K. S. Kedlaya *p-adic Differential Equations*
200. R. L. Frank, A. Laptev & T. Weidl *Schrödinger Operators: Eigenvalues and Lieb–Thirring Inequalities*
201. J. van Neerven *Functional Analysis*
202. A. Schmeding *An Introduction to Infinite-Dimensional Differential Geometry*
203. F. Cabello Sánchez & J. M. F. Castillo *Homological Methods in Banach Space Theory*
204. G. P. Paternain, M. Salo & G. Uhlmann *Geometric Inverse Problems*
205. V. Platonov, A. Rapinchuk & I. Rapinchuk *Algebraic Groups and Number Theory, I (2nd Edition)*
206. D. Huybrechts *The Geometry of Cubic Hypersurfaces*
207. F. Maggi *Optimal Mass Transport on Euclidean Spaces*
208. R. P. Stanley *Enumerative Combinatorics, II (2nd Edition)*
209. M. Kawakita *Complex Algebraic Threefolds*
210. D. Anderson & W. Fulton *Equivariant Cohomology in Algebraic Geometry*
211. G. Pineda Villavicencio *Polytopes and Graphs*

Analytic Combinatorics in Several Variables

Second Edition

ROBIN PEMANTLE
University of Pennsylvania

MARK C. WILSON
University of Massachusetts, Amherst

STEPHEN MELCZER
University of Waterloo, Ontario

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

Shaftesbury Road, Cambridge CB2 8EA, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India

103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of Cambridge University Press & Assessment,
a department of the University of Cambridge.

We share the University's mission to contribute to society through the pursuit of
education, learning and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781108836623

DOI: 10.1017/9781108874144

First published 2013
Second edition 2024

A catalogue record for this publication is available from the British Library

A Cataloging-in-Publication data record for this book is available from the Library of Congress

ISBN 978-1-108-83662-3 Hardback

Cambridge University Press & Assessment
978-1-108-83662-3 — Analytic Combinatorics in Several Variables
Robin Pemantle , Mark C. Wilson , Stephen Melczer
Frontmatter
More Information

To the memory of Philippe Flajolet, on whose shoulders stands all of the work herein.

Contents

Preface to the second edition

There is no perfect time to write a textbook for a field in its infancy. Act too early and the theory might not have the cohesive structure it will eventually develop, but act too late and you might miss an opportunity to encourage new collaborators to enter and shape the area. The first edition of this book was timed to strike a balance between these two extremes: after a sufficient framework for analytic combinatorics in several variables had been developed, but during a time when fundamental results were still being discovered and incorporated.

As a consequence of this choice, the first edition of the text, while influential and put to use by many others in enumerative combinatorics, was presented in a way that many end users found difficult to follow. Having been given the opportunity to create a second edition of this text, after a decade of further development, we are now able to improve both the content and presentation of the field. We have been conscious in this rewriting of making the book more useful for a variety of readers having different motivations, including making it easier to look up and cite desired asymptotic results.

For the second edition, the original authors welcome our active collaborator Stephen Melczer, whose own introductory book on this topic [Mel21] was published recently, and who has rejuvenated the entire enterprise. In contrast to [Mel21], which skips much of the advanced topological and geometric approach to analytic combinatorics in several variables (ACSV) to focus more on elementary arguments and explicit computation, this text remains dedicated to developing the theory in its most general, and most powerful, form. The field of ACSV has flourished since the publication of the first edition, including numerous workshops, seminars, summer school courses, and many publications exploring applications of the theory [Wil15; dALN15; MM16; Pan17; Vid17; Kov19; Mis19; MW19; RWZ20; GE20; Geo21; GFS21; KLM21; GWW21; Len+23].

Over the last ten years, we have gained an improved understanding of the technical parts of the theory. Perhaps the largest change is to give ACSV a rigorous foundation using stratified Morse theory, whereas in the first edition Morse-theoretic arguments were used to motivate constructions that were then verified with other techniques. In addition to fixing numerous typographical and other errors in the first edition, some the fault of the authors and some of the publisher, the following content changes have been made to improve the book.

- The chapters in Parts I (Combinatorial Enumeration) and II (Mathematical Background) have mainly kept their general structure, however much of their discussion has been rewritten. Of particular note, Section 2.4 has been revised to better explain how ACSV for rational functions extends to algebraic functions via diagonal embeddings, and Section 5.4 has been revised to better explain the proof of Theorem 5.3 (formerly Theorem 5.4.8). Chapter 6 in the first edition has also been moved after the former Chapters 7 and 8 so that Part II now ends in Chapter 6 (which was Chapter 7 in the first edition).
- Part III (Multivariate Enumeration) begins with Chapter 7, which has been completely overhauled and is almost entirely new. The second edition is constructed to put the large majority of the topological and homological arguments in this chapter and the appendices. The main output of the chapter is an expression for coefficient asymptotics as a finite integer sum of saddle-point-like integrals, and those wanting to skip the homological material can simply assume this decomposition in later chapters.
- Chapter 8, which is a complete re-imagining of Chapter 6 in the first edition, discusses how to compute the quantities needed for an asymptotic analysis in a computer algebra system. In contrast to the first edition, we now put additional focus on computing the quantities needed for ACSV – this explains its postponement until Part III.
- Chapters 9–11 have been reworked to begin from the decomposition described in Chapter 7, streamlining their presentation, and to have more explicit results that can be easily cited. Section 11.4 has also been expanded to include a worked example of solving a connection problem via creative telescoping.
- The appendices have been revised and enlarged to be more self-contained, and to give readers a more complete explanation of the constructions they will need for ACSV.
- We have greatly increased the number of exercises and examples, and added many more signposts and guides so that readers with different motivations

can find what they are looking for. Exercises have been split into in-text (shorter and more straightforward, meant to help the reader think over the material) and end-of-chapter (more challenging) problems. We have also listed open problems and ongoing research in Chapter 13.

- Finally, we have created Sage worksheets that cover most of the examples in the book, and some of the exercises.

Supplementary material, including Sage worksheets and a maintained list of errata, are available from the book website

`http://acsvproject.org/acsvbook`

More general resources for the ACSV project are available at

`http://acsvproject.org`

The authors thank our colleagues who helped with proofreading and otherwise checking the manuscript, including Nick Beaton, Jeremy Chizewer, Jacob Cordeiro, William Dugan, Stephen Gillen, Kaitian Jin, Alexander Kroitor, Geoffrey Pritchard, Stephan Ramon Garcia, and Josip Smolčić. We thank the anonymous reviewers consulted by the publisher, and in particular one reviewer who set us straight on the inner workings of Thom's Isotopy Theorem. We thank Herman Gluck for help with topology and Frank Sottile for considerable help with computational algebra. A special word of thanks is due to Yuliy Baryshnikov. Not only did we learn most of the recent material from him or with him, but he has remained available for consultation during the entire production of the second edition. The authors each thank their families for their patience and support.

Preface to the first edition

The term "Analytic Combinatorics" refers to the use of complex analytic methods to solve problems in combinatorial enumeration. Its chief objects of study are generating functions [FS09, page vii]. Generating functions have been used for enumeration for over a hundred years, going back to Hardy and, arguably, to Euler. Their systematic study began in the 1950s [Hay56]. Much of the impetus for analytic combinatorics comes from the theory of algorithms, arising for example in the work of Knuth [Knu06]. The recent, seminal work [FS09] describes the rich univariate theory with literally hundreds of applications.

The multivariate theory, as recently as the mid-1990s, was still in its infancy. Techniques for *deriving* multivariate generating functions have been well understood, sometimes paralleling the univariate theory and sometimes achieving surprising depth [FIM99]. Analytic methods for recovering coefficients of generating functions once the functions have been derived have, however, been sorely lacking. A small body of analytic work goes back to the early 1980s [BR83]; however, even by 1995, of 100+ pages in the Handbook of Combinatorics devoted to asymptotic enumeration [Odl95], multivariate asymptotics received fewer than six.

This book is the result of work spanning nearly 15 years. Our aim has been to develop analytic machinery to recover, as effectively as possible, asymptotics of the coefficients of a multivariate generating function. Both authors feel drawn to this area of study because it combines so many areas of modern mathematics. Functions of one or more complex variables are essential, but also algebraic topology in the Russian style, stratified Morse theory, computational algebraic methods, saddle point integration, and of course the basics of combinatorial enumeration. The many applications of this work in areas such as bioinformatics, queueing theory, and statistical mechanics are not surprising when we realize how widespread is the use of generating functions in applied combinatorics and probability.

The purpose of this book is to pass on what we have learned, so that others may learn it and use it before we forget it. The present form of the book grew out of graduate-level mathematics courses that developed, along with the theory, at the University of Wisconsin, Ohio State University, and the University of Pennsylvania. The course was intended to be accessible to students in their second year of graduate study. Because of the eclectic nature of the required background, this presents something of a challenge. One may count on students having seen calculus on manifolds by the end of a year of graduate studies, and some complex variable theory. One may also assume some willingness to do some outside reading. However, some of the more specialized areas on which multivariate analytic combinatorics must draw are not easy to get from books. This includes topics such as the theory of amoebas [GKZ08] and the Leray–Petrovsky–Gårding theory of inverse Fourier transforms. Other topics such as saddle point integration and stratified Morse theory exist in books but require being summarized in order not to cause a semester-long detour.

We have dealt with these problems by summarizing a great amount of background material. Part I contains the combinatorial background and will be known to students who have taken a graduate-level course in combinatorial enumeration. Part II contains mathematical background from outside of combinatorics. The topics in Part II are central to the understanding and execution of the techniques of analytic combinatorics in several variables. Part III contains the theory, all of which is new since the turn of the millennium and only parts of which exist in published form. Finally, there are appendices, almost equal in total size to Part II, which include necessary results from algebraic and differential topology. Some students will have seen these but for the rest, the inclusion of these topics will make the present book self-contained rather than one that can only be read in a library.

We hope to recruit further researchers into this field, which still has many interesting challenges to offer, and this explains the rather comprehensive nature of the book. However, we are aware that some readers will be more focused on applications and seek the solution of a given problem. The book is structured so that after reading Chapter 1, it should be possible to skip to Part III, and pick up supporting material as required from previous chapters. A list of papers using the multivariate methods described in this book can be found on our website: `http://acsvproject.org`.

The mathematical development of the theory belongs mostly to the two authors, but there are a number of individuals whose help was greatly instrumental in moving the theory forward. The complex analysts at the University of Wisconsin-Madison, Steve Wainger, Jean-Pierre Rosay, and Andreas Seeger, helped the authors (then rather junior researchers) to grapple with the problem in its earliest incarnation. A similar role was played several years later by Jeff McNeal. Perhaps the greatest thanks are due to Yuliy Baryshnikov, who translated the Leray–Petrovsky theory and the work of Atiyah–Bott–Gårding into terms the authors could understand, and coauthored several papers. Frank Sottile provided help with algebra on many occasions; Persi Diaconis arranged for a graduate course while the first author visited Stanford in 2000; Richard Stanley answered our numerous miscellaneous queries. Thanks are also due to our other coauthors on papers related to this project, listed on the project website linked from the book website. Alex Raichev and Torin Greenwood helped substantially with proofreading and with computer algebra implementations of some parts of the book. All software can be located via the book website.

On a more personal level, the first author would like to thank his wife, Diana Mutz, for encouraging him to follow this unusual project wherever it took him, even if it meant abandoning a still productive vein of problems in probability theory. The sentiment in the probability theory community may be otherwise, but the many connections of this work to other areas of mathematics have been a source of satisfaction to the authors. The first author would also like to thank his children, Walden, Maria, and Simi, for their participation in the project via the Make-A-Plate company (see Figure 0.1).

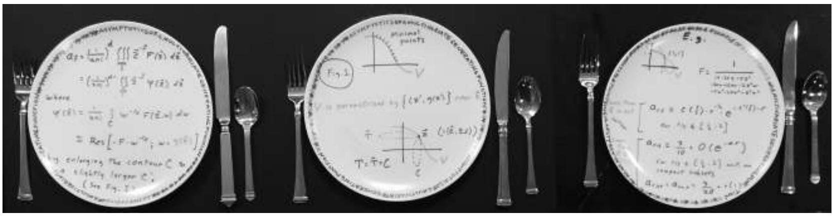

Figure 0.1 Customized "asymptotics of a multivariable generating function" dinner plates.

The second author thanks his wife Golbon Zakeri, children Yusef and Yahya, and mother-in-law Shahin Sabetghadam for their help in carving out time for him to work on this project, sometimes at substantial inconvenience to themselves. He hopes they will agree that the result is worth it.

List of Symbols

$\boldsymbol{r}$	generic multi-index for multivariate array, page 3
$\boldsymbol{z}$	generic element of $\mathbb{C}^d$, page 4
$\lvert\boldsymbol{r}\rvert$	1-norm of multi-index $\boldsymbol{r}$, page 7
$\hat{\boldsymbol{r}}$	unitized vector representing direction determined by $\boldsymbol{r}$, page 7
$\mathbf{1}$	vector with all components equal to 1, page 10
F	generic meromorphic multivariate generating function, page 11
P	numerator of generating function F, page 11
Q	denominator of generating function F, page 11
$\texttt{critical}(\boldsymbol{r})$	set of critical points in direction $\hat{\boldsymbol{r}}$, page 11
$\texttt{contrib}(\boldsymbol{r})$	set of contributing points, page 11
$\Phi_{\boldsymbol{w}}(\boldsymbol{r})$	formula for the contribution from the point $\boldsymbol{w}$ to the asymptotic series for $a_{\boldsymbol{r}}$, page 12
$\boldsymbol{\delta}_j$	vector of length d with a 1 in its jth coordinate and a 0 elsewhere, page 17
$\mathbb{C}[[z_1,\ldots,z_d]]$	ring of formal power series in $\boldsymbol{z} = (z_1,\ldots,z_d)$ with complex coefficients, page 17
$\mathbf{0}$	vector all of whose components equal 0, page 18
∂_k	partial derivative operator with respect to kth variable, page 18
$\mathbb{C}\{z_1,\ldots,z_d\}$	ring of germs of analytic functions, page 19
$\sqcup$	disjoint union, page 22
$\mathbb{P}$	probability measure, page 29
$\mathbb{C}((z))$	field of formal Laurent series, page 42
Re	real part of a complex number, page 77
Im	imaginary part of a complex number, page 77
ϕ	generic phase function of Fourier–Laplace integral, page 89
A	generic amplitude function of Fourier–Laplace integral, page 89
$C(k,\ell)$	constants defined in terms of Gamma function, appearing in Fourier–Laplace integral formulae, page 90

C^M	class of functions whose derivatives to order M are continuous, page 95
C^∞	class of functions whose derivatives of all orders exist, page 95
$\mathfrak{p}$	principal value of kth root, page 98
$\mathcal{I}(\lambda)$	two-sided Fourier–Laplace integral, page 105
$\mathcal{I}_+(\lambda)$	one-sided univariate Fourier–Laplace integral, page 110
Ai	Airy function, page 113
$\mathcal{H}$	Hessian matrix of second partial derivatives, page 114
∇	gradient map, page 114
Relog	coordinatewise log modulus map, page 135
$\mathbb{C}\left[\boldsymbol{z}, \boldsymbol{z}^{-1}\right]$	ring of Laurent polynomials, page 135
$\mathbb{L}(\boldsymbol{z})$	space of formal Laurent expressions, page 135
$\mathtt{amoeba}$	polynomial amoeba, page 141
$\mathtt{hull}$	convex hull, page 142
$\mathcal{N}$	Newton polytope, page 143
ν	order map, page 143
$\mathtt{tan}_{\boldsymbol{x}}(B)$	geometric tangent cone to B at $\boldsymbol{x}$, page 150
$\mathtt{normal}_{\boldsymbol{x}}(B)$	outward normal cone, dual to $\mathtt{tan}_{\boldsymbol{x}}(B)$, page 150
K^*	dual cone of K, page 150
$\mathcal{V}$	singular variety of generating function, page 153
$\nabla_{\log}$	logarithmic gradient, page 159
$\mathcal{C}$	amoeba contour, page 160
deg	order of vanishing of power series, page 162
$\mathtt{algtan}_{\boldsymbol{x}}(f)$	algebraic tangent cone of f at $\boldsymbol{x}$, page 162
$\mathtt{hom}$	homogeneous part of power series, page 162
$\tilde{Q}$	square-free part of Q, page 182
$\mathtt{LT}$	leading term with respect to monomial order, page 223
$\mathbf{L}(\boldsymbol{z})$	logarithmic normal space to the stratum containing $\boldsymbol{z}$, page 229
$\mathcal{G}$	Gauss map, page 281
$\mathcal{K}$	Gaussian curvature, page 281
$M(\mathcal{A})$	matroid of a hyperplane arrangement, page 311
$\mathcal{O}_{\boldsymbol{p}}$	local ring of analytic germs at $\boldsymbol{p}$, page 320
Δ	standard (embedded) simplex, page 330
$\mathbf{t}$	generic variable for simplex, page 330
$\pi\Delta$	shadow simplex, page 330
$\mathcal{S}$	set of critical points for a multiple point Fourier–Laplace integral, page 333
$\mathbf{K}^{\boldsymbol{z}}(A)$	cone of hyperbolicity for the homogeneous polynomial A, page 348
$\mathbf{K}^{A,C}(\boldsymbol{x})$	family of cones for a homogeneous polynomial A, page 349

$\mathbf{K}^{q,B}(\boldsymbol{z})$	family of cones when q is log-Laurent polynomial, page 350
A^*	algebraic dual to the homogeneous polynomial A, page 369
$\mathfrak{n}$	density of standard normal distribution, page 413
δ	coboundary map, page 475
$\mathbf{\Gamma}_\Psi$	augmented lognormal matrix, page 492