1

Acoustics and the Wave Equation

Acoustics is the study of small-amplitude disturbances in a compressible fluid: it is a branch of continuum mechanics. General texts include [711, 644, 696, 207, 227].[1] To simplify our analysis, we suppose that the fluid is inviscid. We derive the governing equations in Section 1.1. Our derivations cover many interesting situations, including inhomogeneous fluids (with properties that can vary spatially and temporally) and non-uniform background flows. However, (at the present time) few of these situations have been combined with scattering phenomena, such as when a sound wave interacts with an object immersed in the flow. Indeed, for most of the book, we shall restrict ourselves to homogeneous fluids. Then, for most (but not all) purposes, it is found that the governing equation is the scalar wave equation; the relevant equations are collected in Section 1.2. Section 1.3 is dedicated to waves on strings, where the motion is governed by the one-dimensional wave equation. For a general survey of the mathematics underlying the wave equation, see the paper by Leis [540]. For another survey, with more emphasis on inverse problems, try [79].

Formal properties of Laplace transforms are collected in Section 1.4, with more detailed discussions reserved for later. The vexed question of *causality* is discussed in Section 1.5. Finally, the governing equations for electromagnetic, elastodynamic and hydrodynamic problems are collected in Section 1.6.

1.1 Governing Equations

The exact equations for the motion of a compressible inviscid fluid are as follows [57, §3.6], [697, §I], [675, §2.1.1]. Conservation of mass gives the continuity equation,

$$\frac{\mathrm{D}\rho_{\mathrm{ex}}}{\mathrm{D}t} + \rho_{\mathrm{ex}}\,\mathrm{div}\,\boldsymbol{v}_{\mathrm{ex}} = 0, \tag{1.1}$$

[1] Citations such as this will be listed in chronological order.

where ρ_{ex} is the mass density, $\boldsymbol{v}_{ex}$ is the fluid velocity and t is time. (The subscript 'ex' denotes 'exact'.) The material derivative is defined by

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial f}{\partial t} + \boldsymbol{v}_{ex} \cdot \operatorname{grad} f.$$

In the absence of body forces, conservation of linear momentum gives

$$\rho_{ex} \frac{\mathrm{D}\boldsymbol{v}_{ex}}{\mathrm{D}t} + \operatorname{grad} p_{ex} = \mathbf{0}, \tag{1.2}$$

where p_{ex} is the pressure. For isentropic flows [57, p. 156], [696, eqn (1-4.3)], the entropy per unit mass, E_{ex}, satisfies

$$\frac{\mathrm{D}E_{ex}}{\mathrm{D}t} = 0. \tag{1.3}$$

There is also an equation of state which we take as the statement that p_{ex} is a function of ρ_{ex} and E_{ex} [696, §1-4],

$$p_{ex} = p_{ex}(\rho_{ex}, E_{ex}). \tag{1.4}$$

Differentiating, we obtain

$$\operatorname{grad} p_{ex} = c_{ex}^2 \operatorname{grad} \rho_{ex} + h_{ex} \operatorname{grad} E_{ex} \tag{1.5}$$

and

$$\frac{\mathrm{D}p_{ex}}{\mathrm{D}t} = c_{ex}^2 \frac{\mathrm{D}\rho_{ex}}{\mathrm{D}t} + h_{ex} \frac{\mathrm{D}E_{ex}}{\mathrm{D}t} = -\rho_{ex} c_{ex}^2 \operatorname{div} \boldsymbol{v}_{ex}, \tag{1.6}$$

using (1.1) and (1.3), where

$$c_{ex}^2(\rho_{ex}, E_{ex}) = \frac{\partial p_{ex}}{\partial \rho_{ex}} \quad \text{and} \quad h_{ex}(\rho_{ex}, E_{ex}) = \frac{\partial p_{ex}}{\partial E_{ex}}. \tag{1.7}$$

Finally, the temperature T_{ex} satisfies [57, eqn (3.6.6)]

$$\frac{1}{T_{ex}} \frac{\mathrm{D}T_{ex}}{\mathrm{D}t} = \frac{\varkappa}{\rho_{ex}} \frac{\mathrm{D}p_{ex}}{\mathrm{D}t} = -\varkappa c_{ex}^2 \operatorname{div} \boldsymbol{v}_{ex}, \tag{1.8}$$

using (1.6), where $\varkappa$ is the ratio of the coefficient of thermal expansion to the specific heat at constant pressure ($\varkappa = \beta/c_p$ in Batchelor's notation [57]).

1.1.1 Linearisation: Ambient Flows

Consider an ambient flow in which $\boldsymbol{v}_{ex} = \boldsymbol{U}$, a constant velocity. (The case $\boldsymbol{U} = \mathbf{0}$ will be of most interest to us.) For such a flow, let $\rho_{ex} = \rho_0$, $p_{ex} = p_0$, $E_{ex} = E_0$, $T_{ex} = T_0$, $c_{ex}^2 = c_0^2$ and $h_{ex} = h_0$. We have $p_0 = p_{ex}(\rho_0, E_0)$, $c_0^2 = c_{ex}^2(\rho_0, E_0)$ and $h_0 = h_{ex}(\rho_0, E_0)$. Then (1.1), (1.2), (1.3), (1.6) and (1.8) give the following constraints on the ambient flow,

$$\frac{\mathscr{D}\rho_0}{\mathscr{D}t} = 0, \quad \operatorname{grad} p_0 = \mathbf{0}, \quad \frac{\mathscr{D}E_0}{\mathscr{D}t} = 0, \quad \frac{\mathscr{D}p_0}{\mathscr{D}t} = 0 \quad \text{and} \quad \frac{\mathscr{D}T_0}{\mathscr{D}t} = 0, \tag{1.9}$$

where $\mathscr{D}f/\mathscr{D}t = \partial f/\partial t + \boldsymbol{U}\cdot\operatorname{grad} f$. Combining $(1.9)_2$ and $(1.9)_4$ shows that p_0 is a constant, whereas (1.5) gives

$$\operatorname{grad} p_0 = c_0^2 \operatorname{grad}\rho_0 + h_0 \operatorname{grad} E_0 = \mathbf{0}. \tag{1.10}$$

The easiest way to satisfy $(1.9)_3$ and $(1.9)_5$ is to suppose that E_0 and T_0 are constants. Then $(1.9)_1$ and (1.10) imply that ρ_0 is constant. In this situation, we say the fluid is *homogeneous*: its properties do not vary with position (or time).

1.1.2 Linearisation: Acoustics

For linear acoustics, we consider small perturbations about the ambient state, and write

$$p_{\text{ex}} = p_0 + \varepsilon p_1 + \cdots, \quad \rho_{\text{ex}} = \rho_0 + \varepsilon\rho_1 + \cdots, \quad \boldsymbol{v}_{\text{ex}} = \boldsymbol{U} + \varepsilon\boldsymbol{v}_1 + \cdots,$$
$$E_{\text{ex}} = E_0 + \varepsilon E_1 + \cdots, \quad c_{\text{ex}} = c_0 + \varepsilon c_1 + \cdots, \quad h_{\text{ex}} = h_0 + \varepsilon h_1 + \cdots,$$

where ε is a small parameter. Substitution in the equation of state (1.4) gives

$$\begin{aligned} p_{\text{ex}}(\rho_{\text{ex}}, E_{\text{ex}}) &= p_{\text{ex}}(\rho_0 + \varepsilon\rho_1 + \cdots, E_0 + \varepsilon E_1 + \cdots) \\ &= p_{\text{ex}}(\rho_0, E_0) + \varepsilon\rho_1 \frac{\partial p_{\text{ex}}}{\partial\rho_{\text{ex}}}(\rho_0, E_0) + \varepsilon E_1 \frac{\partial p_{\text{ex}}}{\partial E_{\text{ex}}}(\rho_0, E_0) + \cdots, \end{aligned}$$

giving $p_0 = p_{\text{ex}}(\rho_0, E_0)$ and

$$p_1 = c_0^2\rho_1 + h_0 E_1 \quad \text{with} \quad c_0^2 = c_{\text{ex}}^2(\rho_0, E_0) \quad \text{and} \quad h_0 = h_{\text{ex}}(\rho_0, E_0).$$

Substitution in (1.1), (1.2), (1.3) and (1.8) gives, at first order in ε,

$$\frac{\mathscr{D}\rho_1}{\mathscr{D}t} + \operatorname{div}(\rho_0\boldsymbol{v}_1) = 0, \quad \rho_0\frac{\mathscr{D}\boldsymbol{v}_1}{\mathscr{D}t} + \operatorname{grad} p_1 = \mathbf{0}, \tag{1.11}$$
$$\frac{\mathscr{D}E_1}{\mathscr{D}t} + \boldsymbol{v}_1\cdot\operatorname{grad} E_0 = 0, \quad \frac{\mathscr{D}T_1}{\mathscr{D}t} + \boldsymbol{v}_1\cdot\operatorname{grad} T_0 = -\varkappa c_0^2 T_0 \operatorname{div}\boldsymbol{v}_1. \tag{1.12}$$

We are mainly interested in perturbations from the ambient state. Therefore we define the excess pressure p by $p_{\text{ex}} = p_0 + p$, and we accept the linear approximation, giving $p = \varepsilon p_1$. We make similar definitions for other relevant quantities. Thus

$$p = p_{\text{ex}} - p_0 = \varepsilon p_1, \quad \boldsymbol{v} = \boldsymbol{v}_{\text{ex}} - \boldsymbol{U} = \varepsilon\boldsymbol{v}_1,$$
$$\tilde{\rho} = \rho_{\text{ex}} - \rho_0 = \varepsilon\rho_1, \quad \widetilde{E} = E_{\text{ex}} - E_0 = \varepsilon E_1, \quad \widetilde{T} = T_{\text{ex}} - T_0 = \varepsilon T_1.$$

The equations relating these quantities are readily found, making use of (1.9). They are

$$p = c_0^2\tilde{\rho} + h_0\widetilde{E}, \quad \frac{\mathscr{D}\tilde{\rho}}{\mathscr{D}t} + \operatorname{div}(\rho_0\boldsymbol{v}) = 0, \quad \rho_0\frac{\mathscr{D}\boldsymbol{v}}{\mathscr{D}t} + \operatorname{grad} p = \mathbf{0}, \tag{1.13}$$
$$\frac{\mathscr{D}\widetilde{E}}{\mathscr{D}t} + \boldsymbol{v}\cdot\operatorname{grad} E_0 = 0, \quad \frac{\mathscr{D}\widetilde{T}}{\mathscr{D}t} + \boldsymbol{v}\cdot\operatorname{grad} T_0 = -\varkappa c_0^2 T_0 \operatorname{div}\boldsymbol{v}. \tag{1.14}$$

These are the basic equations for acoustic small-amplitude perturbations. We examine several special cases below.

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P. A. Martin
Excerpt
More Information

1.1.3 Zero Ambient Velocity: Bergmann's Equation

When $\boldsymbol{U} = \boldsymbol{0}$, (1.9) implies that ρ_0, E_0 and T_0 do not depend on t, whereas p_0 is a constant. The constraint (1.10) permits us to have spatial variations in c_0^2 and ρ_0 within a stationary fluid (but not if E_0 is constant).

For the acoustic perturbation, (1.13) and (1.14) give

$$\frac{\partial \tilde{\rho}}{\partial t} + \operatorname{div}(\rho_0 \boldsymbol{v}) = 0, \quad \rho_0 \frac{\partial \boldsymbol{v}}{\partial t} + \operatorname{grad} p = \boldsymbol{0}, \quad \frac{\partial \widetilde{E}}{\partial t} + \boldsymbol{v} \cdot \operatorname{grad} E_0 = 0. \tag{1.15}$$

As $p = c_0^2 \tilde{\rho} + h_0 \widetilde{E}$ in which c_0^2 and h_0 do not depend on t, we can combine $(1.15)_1$ and $(1.15)_3$ to give

$$\frac{\partial p}{\partial t} + c_0^2 \operatorname{div}(\rho_0 \boldsymbol{v}) + h_0\, \boldsymbol{v} \cdot \operatorname{grad} E_0 = 0. \tag{1.16}$$

Eliminating $h_0 \operatorname{grad} E_0$ using (1.10), we obtain

$$\frac{\partial p}{\partial t} + \rho_0 c_0^2 \operatorname{div} \boldsymbol{v} = 0. \tag{1.17}$$

Finally, eliminating $\boldsymbol{v}$, using the second of (1.15), gives

$$\rho_0 \operatorname{div}\left(\rho_0^{-1} \operatorname{grad} p\right) = \frac{1}{c_0^2} \frac{\partial^2 p}{\partial t^2}, \tag{1.18}$$

in which $\rho_0(\boldsymbol{r})$ and $c_0^2(\boldsymbol{r})$ can be functions of position $\boldsymbol{r} = (x, y, z)$ (but not of t). This is *Bergmann's equation* for the (excess) pressure [97, eqn (14)], [518, eqn (76.1)], [797, eqn (5.15)].

Suppose that the motion is known to be irrotational, meaning that the vorticity $\boldsymbol{\omega} = \operatorname{curl} \boldsymbol{v} = \boldsymbol{0}$. Then we can write $\boldsymbol{v} = \operatorname{grad} u$, where u is a velocity potential. (Note that some authors prefer to write $\boldsymbol{v} = -\operatorname{grad} u$; see, for example, [516, §285] and [644, §6.1].) It follows from $(1.15)_2$ that $p = -\rho_0(\boldsymbol{r})\, \partial u / \partial t$ and then (1.17) yields

$$\nabla^2 u = \frac{1}{c_0^2(\boldsymbol{r})} \frac{\partial^2 u}{\partial t^2}. \tag{1.19}$$

1.1.4 Zero Ambient Velocity and Constant Ambient Density

When $\boldsymbol{U} = \boldsymbol{0}$ and ρ_0 is a constant, Bergmann's equation (1.18) reduces to

$$\nabla^2 p = \frac{1}{c_0^2(\boldsymbol{r})} \frac{\partial^2 p}{\partial t^2}. \tag{1.20}$$

As ρ_0 is constant, taking the curl of $(1.15)_2$ shows that the vorticity $\boldsymbol{\omega} = \operatorname{curl} \boldsymbol{v}$ does not depend on t. Therefore if the motion starts from a state in which $\boldsymbol{v}$ is constant, then $\boldsymbol{\omega} = \boldsymbol{0}$: the motion is irrotational, and we can write $\boldsymbol{v} = \operatorname{grad} u$. Then, as in Section 1.1.3, we have $p = -\rho_0\, \partial u / \partial t$, whereas (1.20) shows that u satisfies the wave equation (1.19).

Note that irrotationality was *assumed* in Section 1.1.3 in order to obtain (1.19), whereas it can be *proved* when ρ_0 is constant.

Equation (1.19) often appears in the context of seismic inversion ('migration'); see, for example, [737], [797, eqn (5.9)]. It also appears in other imaging contexts [720, 521, 653], [326, eqn (2.1)]. Stochastic versions of (1.20), in which $c_0^2(\boldsymbol{r})$ is a random function of position, have also been studied and used; see, for example, [655], [786, eqn (3.17)], [326, eqn (12.1)] and [116].

1.1.5 Zero Ambient Velocity and Homogeneous Fluid

This is the textbook case, in which $\rho_0 \equiv \rho$ and $c_0 \equiv c$ are constants and $\boldsymbol{U} = \boldsymbol{0}$. In most of the book, we shall be concerned with this case.

The governing equations are the wave equation,

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \tag{1.21}$$

for the velocity potential u, together with $p = -\rho\, \partial u/\partial t$ and $\boldsymbol{v} = \operatorname{grad} u$. Evidently, p and any Cartesian component of $\boldsymbol{v}$ also solve the wave equation.

A simpler derivation of the governing equations can be given when the fluid is homogeneous, a derivation in which the entropy does not play a role. We take an equation of state which says that p_{ex} is a function of ρ_{ex}, $p_{\text{ex}} = p_{\text{ex}}(\rho_{\text{ex}})$. Let $\rho_{\text{ex}} = \rho$ and $p_{\text{ex}} = p_0$ when there is no motion, $\boldsymbol{v}_{\text{ex}} = \boldsymbol{0}$. Then (1.1) and (1.2) imply that p_0 is a constant and ρ does not depend on t. Then, in the notation of Section 1.1.2, we find that

$$p_1 = c^2 \rho_1, \quad \text{where} \quad c^2 = p'_{\text{ex}}(\rho) \tag{1.22}$$

is the (constant) speed of sound. Also, from (1.1) and (1.2), we obtain

$$\frac{\partial \rho_1}{\partial t} + \rho \operatorname{div} \boldsymbol{v}_1 = 0 \quad \text{and} \quad \rho \frac{\partial \boldsymbol{v}_1}{\partial t} + \operatorname{grad} p_1 = \boldsymbol{0}. \tag{1.23}$$

Eliminating $\boldsymbol{v}_1$ gives

$$\nabla^2 p_1 = \frac{\partial^2 \rho_1}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 p_1}{\partial t^2}. \tag{1.24}$$

The rest of the derivation, leading to (1.21), proceeds as before. For more details on the derivation of the equations above, see, for example, [644, Chapter 6], [559, Chapter 1], [518, §64] or [696, Chapter 1].

1.1.6 Non-Zero Ambient Velocity and Homogeneous Fluid

In this case, the governing equations are (1.13) and (1.14), in which $\rho_0 \equiv \rho$, $c_0 \equiv c$, h_0 and E_0 are constants:

$$p = c^2 \tilde{\rho} + h_0 \widetilde{E}, \quad \frac{\mathscr{D}\tilde{\rho}}{\mathscr{D}t} + \rho \operatorname{div} \boldsymbol{v} = 0, \quad \frac{\mathscr{D}\widetilde{E}}{\mathscr{D}t} = 0, \quad \rho \frac{\mathscr{D}\boldsymbol{v}}{\mathscr{D}t} + \operatorname{grad} p = \boldsymbol{0}. \tag{1.25}$$

The first three of these give

$$\frac{\mathscr{D}p}{\mathscr{D}t} = -\rho c^2 \operatorname{div} \boldsymbol{v}$$

from which we can eliminate $\boldsymbol{v}$ using $(1.25)_4$ to obtain

$$\nabla^2 p = \frac{1}{c^2}\frac{\mathscr{D}^2 p}{\mathscr{D}t^2} = \frac{1}{c^2}\left(\frac{\partial}{\partial t} + \boldsymbol{U}\cdot\operatorname{grad}\right)^2 p. \tag{1.26}$$

This is the *convected wave equation* [697, eqn (6)], [421, eqn (1.6.30)]. If the flow is irrotational, with $\boldsymbol{v} = \operatorname{grad} u$, we find that the potential u also satisfies (1.26) with $p = -\rho(\partial u/\partial t + \boldsymbol{U}\cdot\operatorname{grad} u)$. Applications of (1.26) will be mentioned in Section 7.5.

As the fluid is homogeneous and $\boldsymbol{U}$ is a constant vector, we can also obtain (1.26) using a Galilean transformation. Thus, if $(\boldsymbol{r}, t)$ is a fixed frame in which the ambient velocity is $\boldsymbol{U}$, introduce a translating frame $(\boldsymbol{r}', t')$ with $\boldsymbol{r} = \boldsymbol{r}' + \boldsymbol{U}t'$ and $t = t'$. The chain rule gives, for example, $\partial u/\partial x' = \partial u/\partial x$, $\partial^2 u/\partial x'^2 = \partial^2 u/\partial x^2$ and $\partial u/\partial t' = \partial u/\partial t + \boldsymbol{U}\cdot\operatorname{grad} u$. Hence, if u satisfies the wave equation with $\boldsymbol{r}' = (x', y', z')$ and t' as independent variables, then u satisfies the convected wave equation with independent variables $\boldsymbol{r} = (x, y, z)$ and t.

Equation (1.26) was used by Tatarski [814, eqn (5.1)] with $\boldsymbol{U}$ replaced by $\boldsymbol{U}(\boldsymbol{r})$, the local ambient velocity at position $\boldsymbol{r}$; see also [697, eqn (4)]. There are other versions of the convected wave equation that are intended for inhomogeneous fluids with a non-uniform ambient flow; see [697, 675, 146] and Section 1.1.7.

1.1.7 Non-Uniform Ambient Flows and Dynamic Materials

Sound transmission through a fluctuating ocean [302] exemplifies a problem in which the ambient flow is non-uniform in both space and time. We have already seen examples in which the background medium varies spatially (Section 1.1.3) but not in time. Media with temporal variations may be called *dynamic materials*. Such materials arise naturally (the oceans and the atmosphere are obvious examples) but there is also a growing interest in their creation. For some background and many applications, see [723, 582].

Perhaps the simplest model of dynamic materials is obtained by allowing c to be a function of time, giving [157]

$$\nabla^2 w = \frac{1}{c^2(t)}\frac{\partial^2 w}{\partial t^2}. \tag{1.27}$$

More generally, models of the form $\operatorname{div}\{a(\boldsymbol{r}, t)\operatorname{grad} w\} = \partial^2 w/\partial t^2$ have been used [585]. In such models, including (1.27), no physical meaning is attributed to w. For some related one-dimensional studies, see [292, 755, 2, 891].

Pierce's Equation

For acoustic problems, Pierce [697] has derived a Bergmann-like wave equation, under certain assumptions about the dynamic medium: he assumes that it is 'slowly varying with position over distances comparable to a representative acoustic wavelength and that it is slowly varying with time over times comparable to a representative acoustic period' [697, p. 2293]. A stochastic version of Pierce's equation has been used recently [117].

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For zero ambient velocity ($\boldsymbol{U} = \mathbf{0}$), Pierce's equation [697, eqn (23)] reduces to

$$\frac{1}{\rho_0(\boldsymbol{r})}\operatorname{div}\{\rho_0(\boldsymbol{r})\operatorname{grad} u\} = \frac{\partial}{\partial t}\left(\frac{1}{c_0^2(\boldsymbol{r},t)}\frac{\partial u}{\partial t}\right), \tag{1.28}$$

where $u(\boldsymbol{r},t)$ is a velocity potential: $\boldsymbol{v} = \operatorname{grad} u$ and $p = -\rho_0\,\partial u/\partial t$. Equation (1.28) is W3 in the collection compiled by Campos [146]. Flatté [302, eqn (5.1.11) with eqn (6.1.1)] uses another equation for u,

$$\nabla^2 u = \frac{1}{c_0^2(\boldsymbol{r},t)}\frac{\partial^2 u}{\partial t^2}, \tag{1.29}$$

which reduces to Bergmann's equation (1.19) when c_0^2 does not depend on t. It has been remarked that the 'apparent simplicity of linearity [in (1.29)] is superseded by the complexity brought in by space-time inhomogeneity and [is] pregnant of exotic wave-like effects' [723, p. 928].

Note that we have written $\rho_0(\boldsymbol{r})$ in (1.28), not $\rho_0(\boldsymbol{r},t)$. This is because we showed in Section 1.1.3 that conservation of mass combined with $\boldsymbol{U} = \mathbf{0}$ implies that ρ_0 cannot depend on t. In other words, if we want to have $\rho_0(\boldsymbol{r},t)$, then we must have a moving ambient flow or we must abandon conservation of mass.

Note also that if c_0^2 does not depend on t, then (1.28) does not reduce to Bergmann's equation (1.19). Pierce [697, eqn (30)] attributes the discrepancy to a second-order effect that may be discarded.

In [39], the authors model a dynamic material by modifying Bergmann's equation (1.18), which we write as

$$\operatorname{div}\left(\frac{1}{\rho_0(\boldsymbol{r})}\operatorname{grad} p\right) = \kappa_0(\boldsymbol{r})\frac{\partial^2 p}{\partial t^2}, \tag{1.30}$$

in which $\kappa_0 = (\rho_0 c_0^2)^{-1}$ is the (adiabatic) *compressibility* [696, p. 30]. In [39, eqn (4)], (1.30) is used but with $\rho_0(\boldsymbol{r},t)$ in place of $\rho_0(\boldsymbol{r})$. We have seen that such an equation is inconsistent with conservation of mass. This provides one motivation for relaxing the constraint of mass conservation. Another comes from continuum models of growing materials [349, Part IV], [260].

Exponential Growth

Let us abandon conservation of mass, replacing (1.1) by

$$\frac{\mathrm{D}\rho_{\mathrm{ex}}}{\mathrm{D}t} + \rho_{\mathrm{ex}}\operatorname{div}\boldsymbol{v}_{\mathrm{ex}} = \rho_{\mathrm{ex}}\gamma, \tag{1.31}$$

where $\gamma(\boldsymbol{r})$ is a given function of position, the *growth rate function*; for this model, see [349, eqn (13.5)]. We retain the other governing equations, namely (1.2), (1.3) and (1.4).

Linearising about an ambient state in which $\boldsymbol{U} = \mathbf{0}$ (and ignoring any temperature dependence), we find that p_0 is constant, E_0 does not depend on t and

$$\frac{\partial\rho_0}{\partial t} = \rho_0\gamma \quad \text{whence} \quad \rho_0(\boldsymbol{r},t) = \rho_{00}(\boldsymbol{r})\,\mathrm{e}^{t\gamma(\boldsymbol{r})}, \tag{1.32}$$

where $\rho_{00}(\boldsymbol{r}) = \rho_0(\boldsymbol{r},0)$. As (1.10) also holds, we substitute ρ_0 and obtain

$$\mu\,\mathrm{e}^{\gamma t}\,(\operatorname{grad}\rho_{00} + t\rho_{00}\operatorname{grad}\gamma) + \operatorname{grad}E_0 = \mathbf{0}$$

where $\mu(\boldsymbol{r},t) = c_0^2/h_0$. To eliminate the term containing $t\rho_{00}$, we are forced to take $\operatorname{grad}\gamma(\boldsymbol{r}) = \mathbf{0}$: γ is a constant, γ_0, say. Then, we infer that $\mu\,\mathrm{e}^{\gamma_0 t}$ cannot depend on t, whence

$$\mu(\boldsymbol{r},t) = \mu_0(\boldsymbol{r})\,\mathrm{e}^{-\gamma_0 t} \tag{1.33}$$

and $\kappa_0^{-1} = \rho_0 c_0^2 = \rho_0\mu h_0 = \rho_{00}(\boldsymbol{r})\mu_0(\boldsymbol{r})h_{\mathrm{ex}}(\rho_0(\boldsymbol{r},t),E_0(\boldsymbol{r}))$, which depends on t, in general.

For the acoustic perturbation, we obtain a slightly modified form of (1.15):

$$\frac{\partial\tilde{\rho}}{\partial t} + \operatorname{div}(\rho_0\boldsymbol{v}) = \tilde{\rho}\gamma_0, \qquad \rho_0\frac{\partial\boldsymbol{v}}{\partial t} + \operatorname{grad}p = \mathbf{0}, \tag{1.34}$$

$$p = c_0^2\tilde{\rho} + h_0\widetilde{E}, \qquad \frac{\partial\widetilde{E}}{\partial t} + \boldsymbol{v}\cdot\operatorname{grad}E_0 = 0. \tag{1.35}$$

As E_0 does not depend on t, differentiating $(1.35)_2$ gives

$$\frac{\partial^2\widetilde{E}}{\partial t^2} = -\frac{\partial\boldsymbol{v}}{\partial t}\cdot\operatorname{grad}E_0 = \frac{1}{\rho_0}(\operatorname{grad}p)\cdot(\operatorname{grad}E_0)$$

after use of $(1.34)_2$. Eliminating $\widetilde{E}$ using $(1.35)_1$ and $\operatorname{grad}E_0$ using (1.10), we arrive at

$$\frac{h_0}{c_0^2}\frac{\partial^2}{\partial t^2}\left(\frac{p - c_0^2\tilde{\rho}}{h_0}\right) = -\frac{1}{\rho_0}(\operatorname{grad}p)\cdot(\operatorname{grad}\rho_0), \tag{1.36}$$

which is an equation relating p and $\tilde{\rho}$.

For a second equation, we start by integrating $(1.34)_2$. Let $\boldsymbol{g}(\boldsymbol{r},t) = \rho_0^{-1}\operatorname{grad}p$ and suppose that $\boldsymbol{v}(\boldsymbol{r},0) = \mathbf{0}$. Then

$$\boldsymbol{v}(\boldsymbol{r},t) = -\int_0^t \boldsymbol{g}(\boldsymbol{r},\tau)\,\mathrm{d}\tau. \tag{1.37}$$

We substitute this expression in $(1.34)_1$:

$$\frac{\partial\tilde{\rho}}{\partial t} - \tilde{\rho}\gamma_0 = \operatorname{div}\left(\rho_0(\boldsymbol{r},t)\int_0^t \boldsymbol{g}(\boldsymbol{r},\tau)\,\mathrm{d}\tau\right) = F(\boldsymbol{r},t), \tag{1.38}$$

say. Assuming that $\tilde{\rho}(\boldsymbol{r},0) = 0$, we can solve for $\tilde{\rho}$:

$$\tilde{\rho}(\boldsymbol{r},t) = \int_0^t \mathrm{e}^{\gamma_0(t-\tau')}F(\boldsymbol{r},\tau')\,\mathrm{d}\tau'. \tag{1.39}$$

Hence $\partial\tilde{\rho}/\partial t = F + \gamma_0\tilde{\rho}$ and $\partial^2\tilde{\rho}/\partial t^2 = \partial F/\partial t + \gamma_0 F + \gamma_0^2\tilde{\rho}$. Using these relations, we substitute (1.39) in (1.36), recalling that $\mu(\boldsymbol{r},t) = c_0^2/h_0$ is given by (1.33):

$$\frac{1}{\mu}\frac{\partial^2(\mu\tilde{\rho})}{\partial t^2} = \frac{\partial^2\tilde{\rho}}{\partial t^2} + \frac{2}{\mu}\frac{\partial\mu}{\partial t}\frac{\partial\tilde{\rho}}{\partial t} + \frac{\tilde{\rho}}{\mu}\frac{\partial^2\mu}{\partial t^2} = \frac{\partial F}{\partial t} - \gamma_0 F. \tag{1.40}$$

Next, let us evaluate F, defined by (1.38). Making use of $(1.32)_2$,

$$F(\boldsymbol{r},t) = \operatorname{div}\int_0^t \mathrm{e}^{\gamma_0(t-\tau)}\operatorname{grad}p(\boldsymbol{r},\tau)\,\mathrm{d}\tau = \int_0^t \mathrm{e}^{\gamma_0(t-\tau)}\,\nabla^2 p(\boldsymbol{r},\tau)\,\mathrm{d}\tau.$$

Hence

$$\frac{\partial F}{\partial t} = \nabla^2 p + \gamma_0 \int_0^t \mathrm{e}^{\gamma_0(t-\tau)}\,\nabla^2 p\,\mathrm{d}\tau = \nabla^2 p + \gamma_0 F. \tag{1.41}$$

Using (1.40) and (1.41), (1.36) becomes

$$\frac{h_0}{c_0^2}\frac{\partial^2}{\partial t^2}\left(\frac{p}{h_0}\right) = \frac{1}{\mu}\frac{\partial^2(\mu\tilde{\rho})}{\partial t^2} - \frac{1}{\rho_0}(\operatorname{grad}p)\cdot(\operatorname{grad}\rho_0) = \rho_0\operatorname{div}\left(\frac{\operatorname{grad}p}{\rho_0}\right). \tag{1.42}$$

Evidently, this is a generalisation of Bergmann's equation (1.18).

More General Growth Models

We have seen that if we start from (1.31) with growth rate function $\gamma(\boldsymbol{r})$, then we are forced to take $\gamma = \gamma_0$, a constant, so that spatial variation of γ is lost. For a more general model, we could replace (1.31) with

$$\frac{\mathrm{D}\rho_{\mathrm{ex}}}{\mathrm{D}t} + \rho_{\mathrm{ex}}\operatorname{div}\boldsymbol{v}_{\mathrm{ex}} = \rho_{\mathrm{ex}}\frac{\partial\eta}{\partial t}, \tag{1.43}$$

where $\eta(\boldsymbol{r},t)$ is specified. Proceeding as with the model (1.31), it turns out that $p(\boldsymbol{r},t)$ satisfies a complicated integrodifferential equation; see [604] for details.

Further growth models could be developed. Notice that the model (1.43) is simple (and linear in ρ_{ex}), so there is plenty of scope for alternative models.

No Growth Model at All: Specify the Background Density

Instead of replacing conservation of mass by a growth model, such as (1.31) or (1.43), let us simply specify $\rho_0(\boldsymbol{r},t)$, assuming that this specification is contrived by some external means. This is a plausible approach if we wish to create dynamic materials. As before, we take $\boldsymbol{U} = \boldsymbol{0}$, and we find that p_0 is constant and $\partial E_0/\partial t = 0$. Then, from (1.6), we obtain

$$0 = \frac{\partial p_0}{\partial t} = c_0^2\frac{\partial\rho_0}{\partial t} + h_0\frac{\partial E_0}{\partial t},$$

which reduces to $\partial\rho_0/\partial t = 0$. In other words, if we want $\partial\rho_0/\partial t \neq 0$, then we must modify (1.3), $\mathrm{D}E_{\mathrm{ex}}/\mathrm{D}t = 0$. This could be done, perhaps by retaining temperature effects [57, eqn (3.6.3)], [696, eqn (1-4.6)]. However, as far as we know, this option has not been contemplated.

Final Comments

The discussion in this section is essentially exact, within the limits of perturbation theory. We have not introduced additional approximations, such as those arising from relevant time scales. For example, the time scale associated with acoustic disturbances is much shorter than those associated with biological growth [349, §13.1].

Nevertheless, we should keep in mind that technological progress may lead to dynamic materials that can change rapidly, thereby making material and acoustic time scales comparable.

1.1.8 Nonlinear Acoustics

The linear equations derived above are sufficient for most of what follows. However, occasionally, an exact formulation is needed. If we restrict to flows that are irrotational ($\operatorname{curl}\boldsymbol{v}_{\text{ex}} = \mathbf{0}$) and homentropic ($E_{\text{ex}}$ is constant), an exact equation for the exact velocity potential u_{ex} can be derived,

$$\mathscr{C}^2\nabla^2 u_{\text{ex}} - \frac{\partial^2 u_{\text{ex}}}{\partial t^2} = \frac{\partial v_{\text{ex}}^2}{\partial t} + \frac{1}{2}\boldsymbol{v}_{\text{ex}}\cdot\operatorname{grad} v_{\text{ex}}^2, \tag{1.44}$$

where $\boldsymbol{v}_{\text{ex}} = \operatorname{grad} u_{\text{ex}}$ and $v_{\text{ex}} = |\boldsymbol{v}_{\text{ex}}|$. The quantity $\mathscr{C}^2$ depends on the fluid and the flow. For a polytropic gas with ratio of specific heats γ, we have

$$\mathscr{C}^2 = c_{\text{ex}}^2 - (\gamma - 1)\left(\frac{\partial u_{\text{ex}}}{\partial t} + \frac{1}{2}v_{\text{ex}}^2\right), \tag{1.45}$$

where c_{ex}^2 is the usual speed of sound, (1.7). Substituting (1.45) and $\boldsymbol{v}_{\text{ex}} = \operatorname{grad} u_{\text{ex}}$ in (1.44) gives a complicated nonlinear partial differential equation for the velocity potential u_{ex}. For a derivation of (1.44) and (1.45), see [384, §3.2] or [161, §4.4].

Equations (1.44) and (1.45) were written down in a paper by Longhorn [569, §6]. An equation similar to (1.44) can be found in [108, eqn (1.85)]. Equations (1.44) and (1.45) provide a firm foundation for quantifying nonlinear effects arising from inviscid, irrotational, compressible flows generated by the motions of spherical objects, for example; see [569, 317, 544]. For direct numerical simulation of such flows, see [683]. For nonlinear acoustics generally, see [383, 257], for example.

1.2 Acoustic Scattering

We have seen (Section 1.1.5) that linear acoustics in a homogeneous inviscid compressible fluid is governed by the wave equation. In three dimensions, this equation is

$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \tag{1.46}$$

where x, y and z are Cartesian coordinates, t is time and c is the (positive) constant speed of sound. We always consider u to be a velocity potential, so that the velocity and (excess) pressure in the fluid are given by

$$\boldsymbol{v} = \operatorname{grad} u \quad \text{and} \quad p = -\rho\frac{\partial u}{\partial t}, \tag{1.47}$$

respectively, where ρ is the constant ambient density of the fluid. Solutions of (1.46) are called *wavefunctions*.