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Representing more than 15 years of work, this will be the leading text on the subject for years to come.

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Cambridge University Press & Assessment
 978-1-108-83546-6 — Initial Algebras and Terminal Coalgebras
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Initial Algebras and Terminal Coalgebras

The Theory of Fixed Points of Functors

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Cambridge University Press & Assessment
 978-1-108-83546-6 — Initial Algebras and Terminal Coalgebras
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Shaftesbury Road, Cambridge CB2 8EA, United Kingdom
 One Liberty Plaza, 20th Floor, New York, NY 10006, USA
 477 Williamstown Road, Port Melbourne, VIC 3207, Australia
 314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre,
 New Delhi – 110025, India
 103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of Cambridge University Press & Assessment,
 a department of the University of Cambridge.

We share the University's mission to contribute to society through the pursuit of
 education, learning and research at the highest international levels of excellence.

www.cambridge.org
 Information on this title: www.cambridge.org/9781108835466

DOI: 10.1017/9781108884112

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 of relevant collective licensing agreements, no reproduction of any part may take
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When citing this work, please include a reference to the DOI 10.1017/9781108884112

First published 2025

A catalogue record for this publication is available from the British Library

Library of Congress Cataloging-in-Publication Data

Names: Adámek, Jiří, 1947– author. | Milius, Stefan, author. |
 Moss, Lawrence Stuart, 1959– author.

Title: Initial algebras and terminal coalgebras : the theory of fixed points
 of functors / Jiří Adámek (Czech Technical University in Prague),
 Stefan Milius (Friedrich-Alexander-Universität Erlangen-Nürnberg, Germany),
 Lawrence S. Moss (Indiana University, Bloomington).

Description: Cambridge, United Kingdom ; New York, NY : Cambridge
 University Press, 2025. | Series: Cambridge tracts in theoretical
 computer science ; 62 | Includes bibliographical references and index.

Identifiers: LCCN 2024011074 | ISBN 9781108835466 (hardback) |
 ISBN 9781108884112 (ebook)

Subjects: LCSH: Categories (Mathematics) | Recursion theory. |
 Induction (Mathematics) | Fixed point theory. |
 Duality theory (Mathematics)

Classification: LCC QA169 .A31994 2025 | DDC 512/.62–dc23/eng/20240628
 LC record available at <https://lcn.loc.gov/2024011074>

ISBN 978-1-108-83546-6 Hardback

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To the memory of Věra Trnková.

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Preface

Initial algebras for endofunctors on a category have been used since the 1970s in algebraic specification and for the semantics of inductive data types definitions. They provide a generic framework to study notions such as recursive function definitions and proofs by structural induction. This has been developed for example in the work of the ADJ group in the 1970s on initial-algebra semantics of abstract data types [143]. Domain theory is another example. Dana Scott's model of the lambda calculus [273] works with initial algebras for endofunctors on the category of domains or complete partial orders [138]. This usage developed into Michael Smyth and Gordon Plotkin's treatment of the solution of recursive domain equations [279]. The 1980s and 90s saw the further development of such topics and their treatment in textbooks such as Ernest Manes and Michael Arbib's book [220] on algebraic semantics of programming, and Samson Abramsky and Achim Jung's survey of domain theory [2].

At the turn of the new millennium, the dual concept, coalgebras for endofunctors, attracted increased attention. While the study of coalgebras had its roots in earlier work parallel to algebras, it was sparked in earnest by Peter Aczel's book [3] on non-well-founded sets, where coalgebras were mentioned at the end. His observation was that the infinite processes that we see in theoretical computer science may be profitably studied as elements of the terminal coalgebra, with specifications of them coming from other coalgebras. He and Nax Mendler also exhibited Robin Milner and David Park's notion of bisimulation from process algebra as a coalgebraic notion. This then led to the breakthrough essay on coalgebras as a theory of systems, Jan Rutten's seminal paper [264]. It demonstrated that many types of state-based systems studied in fields such as automata theory, concurrency, and verification arise as examples of coalgebras for endofunctors. Moreover, the terminal coalgebra yields a fully abstract domain for the behaviour of states of systems. This laid the basis for the new subject of *universal coalgebra*. In the 2000s the subject then rapidly developed and

has unified a host of topics that looked similar but were not always understood that way. These were topics from theoretical computer science and logic like automata theory, process calculi, streams, non-well-founded sets, and modal logic, and also areas of mathematics such as power series. Meanwhile universal coalgebra has become a diverse research field offering a generic framework for the semantics of state-based systems, specification and proof principles such as corecursion and coinduction, and the development of coalgebraic logics. In the last few years, generic methods and algorithms for reasoning, model checking, minimization and learning of coalgebras have become a focus of research. Coalgebra continues to be a lively and active area of research, and we hope to offer our readers a source that will help them to enter the field.

Throughout this development it has been the pull of category theory that has provided the language and conceptual apparatus that is needed to unify topics and pose new questions. Therefore, the aim of our book is to give a category-theoretic account of initial algebras, terminal coalgebras, and, as the title of our book suggests, to pursue the topic of fixed points of functors more generally. We also put a focus on the interplay of algebras and coalgebras, a feature that most other texts on those subjects miss, treating one or the other only. Many of the results in this book are stated and proved for categories that go beyond the category of sets, e.g. complete posets and complete metric spaces. Furthermore, we use special features of the category of sets such as presentations of finitary functors and transfinite recursion. We frequently call on facts about set functors, so much so that we have decided to provide an appendix on this topic. A number of these facts are based on work by the Prague mathematical community, especially Věra Trnková and her colleagues and students Václav Koubek and Jan Reiterman. To make this more ‘accessible’ for readers not so familiar with it, we provide some background on those topics as needed.

The central topics of our book are:

- (1) The iterative constructions of initial algebras and terminal algebras as colimits of chains and limits of op-chains, respectively. We discuss this at great length, providing hosts of concrete examples and developing generalizations to the transfinite setting. We also develop the enriched setting, in particular, enrichment over the categories of complete partial orders and of complete metric spaces, that has been so useful in topics like domain theory.
- (2) Initial algebras and terminal coalgebras of finitary functors, and accessible functors more generally. For example, we develop and extend the work of James Worrell [316, 318] on the terminal coalgebra of accessible set functors.
- (3) Categorical recursion theory, especially corecursion, completely iterative algebras and well-founded coalgebras. This not only connects our subject to the

past but also highlights topics that we believe will be ever more important in the future.

(4) The relations between various fixed points. In particular, we treat the *rational fixed point* of a functor, which is a fully abstract domain of ‘finite-state’ behaviour with instances such as the regular languages from automata theory. Here again we go beyond sets for most results.

It has been said that there never is a good time to write a book about anything, and this is especially true of a book coming from an active research field. We are not only trying to summarize algebra, coalgebra and related fields, we are also contributing to them. Some of the results in this book appear here for the first time, and in many other cases we have revised our text in order to develop a subject that only came into existence a few years ago. We hope that an up-to-the-moment book itself will prove useful to the reader.

We have tried hard to be scholarly about the history of results, and we hope that those whose work was not mentioned correctly, or at all, will forgive us.

We have neglected several topics that are near and dear to the hearts of many in the coalgebra community. We did this partly to keep the book of reasonable length, and partly because those topics are already treated in books. In particular, we are thinking of bisimulation and of coalgebraic modal logic. Bisimulation is treated in Bart Jacobs’s 2016 book [173]; that book also has a lot of material on another topic which we do not treat: predicate liftings. Coalgebraic modal logic is the topic of Dirk Pattinson and Lutz Schröder’s forthcoming book. In addition, one also should read Jan Rutten’s *The Method of Coalgebra: Exercises in Coinduction* [268]. These are all excellent resources, and they will be especially useful for newcomers to the subject.

But now we hope our readers will have as much fun with this book as we have had in writing it.

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