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Initial Algebras and Terminal Coalgebras

Providing an in-depth treatment of an exciting research area, this text's central topics are initial algebras and terminal coalgebras, primary objects of study in all areas of theoretical computer science connected to semantics.

It contains a thorough presentation of iterative constructions, giving both classical and new results on terminal coalgebras obtained by limits of canonical chains and initial algebras obtained by colimits. These constructions are also developed in enriched settings, especially those enriched over complete partial orders and over complete metric spaces, connecting the book to topics such as domain theory. Also included are an extensive treatment of set functors and the first book-length presentation of the rational fixed point of a functor and of lifting results that connect fixed points of set functors with fixed points of endofunctors on other categories.

Representing more than 15 years of work, this will be the leading text on the subject for years to come.

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Initial Algebras and Terminal Coalgebras

The Theory of Fixed Points of Functors

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To the memory of Věra Trnková.

Contents

Preface

Initial algebras for endofunctors on a category have been used since the 1970s in algebraic specification and for the semantics of inductive data types definitions. They provide a generic framework to study notions such as recursive function definitions and proofs by structural induction. This has been developed for example in the work of the ADJ group in the 1970s on initial-algebra semantics of abstract data types [143]. Domain theory is another example. Dana Scott's model of the lambda calculus [273] works with initial algebras for endofunctors on the category of domains or complete partial orders [138]. This usage developed into Michael Smyth and Gordon Plotkin's treatment of the solution of recursive domain equations [279]. The 1980s and 90s saw the further development of such topics and their treatment in textbooks such as Ernest Manes and Michael Arbib's book [220] on algebraic semantics of programming, and Samson Abramsky and Achim Jung's survey of domain theory [2].

At the turn of the new millennium, the dual concept, coalgebras for endofunctors, attracted increased attention. While the study of coalgebras had its roots in earlier work parallel to algebras, it was sparked in earnest by Peter Aczel's book [3] on non-well-founded sets, where coalgebras were mentioned at the end. His observation was that the infinite processes that we see in theoretical computer science may be profitably studied as elements of the terminal coalgebra, with specifications of them coming from other coalgebras. He and Nax Mendler also exhibited Robin Milner and David Park's notion of bisimulation from process algebra as a coalgebraic notion. This then led to the breakthrough essay on coalgebras as a theory of systems, Jan Rutten's seminal paper [264]. It demonstrated that many types of state-based systems studied in fields such as automata theory, concurrency, and verification arise as examples of coalgebras for endofunctors. Moreover, the terminal coalgebra yields a fully abstract domain for the behaviour of states of systems. This laid the basis for the new subject of *universal coalgebra*. In the 2000s the subject then rapidly developed and has unified a host of topics that looked similar but were not always understood that way. These were topics from theoretical computer science and logic like automata theory, process calculi, streams, non-well-founded sets, and modal logic, and also areas of mathematics such as power series. Meanwhile universal coalgebra has become a diverse research field offering a generic framework for the semantics of state-based systems, specification and proof principles such as corecursion and coinduction, and the development of coalgebraic logics. In the last few years, generic methods and algorithms for reasoning, model checking, minimization and learning of coalgebras have become a focus of research. Coalgebra continues to be a lively and active area of research, and we hope to offer our readers a source that will help them to enter the field.

Throughout this development it has been the pull of category theory that has provided the language and conceptual apparatus that is needed to unify topics and pose new questions. Therefore, the aim of our book is to give a category-theoretic account of initial algebras, terminal coalgebras, and, as the title of our book suggests, to pursue the topic of fixed points of functors more generally. We also put a focus on the interplay of algebras and coalgebras, a feature that most other texts on those subjects miss, treating one or the other only. Many of the results in this book are stated and proved for categories that go beyond the category of sets, e.g. complete posets and complete metric spaces. Furthermore, we use special features of the category of sets such as presentations of finitary functors and transfinite recursion. We frequently call on facts about set functors, so much so that we have decided to provide an appendix on this topic. A number of these facts are based on work by the Prague mathematical community, especially Věra Trnková and her colleagues and students Vacláv Koubek and Jan Reiterman. To make this more ‘accessible’ for readers not so familiar with it, we provide some background on those topics as needed.

The central topics of our book are:

(1) The iterative constructions of initial algebras and terminal algebras as colimits of chains and limits of op-chains, respectively. We discuss this at great length, providing hosts of concrete examples and developing generalizations to the transfinite setting. We also develop the enriched setting, in particular, enrichment over the categories of complete partial orders and of complete metric spaces, that has been so useful in topics like domain theory.

(2) Initial algebras and terminal coalgebras of finitary functors, and accessible functors more generally. For example, we develop and extend the work of James Worrell [316, 318] on the terminal coalgebra of accessible set functors.

(3) Categorical recursion theory, especially corecursion, completely iterative algebras and well-founded coalgebras. This not only connects our subject to the

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past but also highlights topics that we believe will be ever more important in the future.

(4) The relations between various fixed points. In particular, we treat the *rational fixed point* of a functor, which is a fully abstract domain of 'finite-state' behaviour with instances such as the regular languages from automata theory. Here again we go beyond sets for most results.

It has been said that there never is a good time to write a book about anything, and this is especially true of a book coming from an active research field. We are not only trying to summarize algebra, coalgebra and related fields, we are also contributing to them. Some of the results in this book appear here for the first time, and in many other cases we have revised our text in order to develop a subject that only came into existence a few years ago. We hope that an up-to-the-moment book itself will prove useful to the reader.

We have tried hard to be scholarly about the history of results, and we hope that those whose work was not mentioned correctly, or at all, will forgive us.

We have neglected several topics that are near and dear to the hearts of many in the coalgebra community. We did this partly to keep the book of reasonable length, and partly because those topics are already treated in books. In particular, we are thinking of bisimulation and of coalgebraic modal logic. Bisimulation is treated in Bart Jacob's 2016 book [173]; that book also has a lot of material on another topic which we do not treat: predicate liftings. Coalgebraic modal logic is the topic of Dirk Pattinson and Lutz Schröder's forthcoming book. In addition, one also should read Jan Rutten's *The Method of Coalgebra: Exercises in Coinduction* [268]. These are all excellent resources, and they will be especially useful for newcomers to the subject.

But now we hope our readers will have as much fun with this book as we have had in writing it.

Jiří Adámek
Stefan Milius
Lawrence S. Moss