Foundations of Constructive Probability Theory

Using Bishop's work on constructive analysis as a framework, this monograph gives a systematic, detailed, and general constructive theory of probability theory and stochastic processes. It is the first extended account of this theory: Almost all of the constructive existence and continuity theorems that permeate the book are original. It also contains results and methods hitherto unknown in the constructive and nonconstructive settings. The text features logic only in the common sense and, beyond a certain mathematical maturity, requires no prior training in either constructive mathematics or probability theory. It will thus be accessible and of interest to both probabilists interested in the foundations of their specialty and constructive mathematicians who wish to see Bishop's theory applied to a particular field.

Yuen-Kwok Chan completed a PhD in constructive mathematics with Errett Bishop before leaving academia for a career in private industry. He is now an independent researcher in probability and its applications.

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

This series is devoted to significant topics or themes that have wide application in mathematics or mathematical science and for which a detailed development of the abstract theory is less important than a thorough and concrete exploration of the implications and applications.

Books in the **Encyclopedia of Mathematics and Its Applications** cover their subjects comprehensively. Less important results may be summarized as exercises at the ends of chapters. For technicalities, readers can be referred to the bibliography, which is expected to be comprehensive. As a result, volumes are encyclopedic references or manageable guides to major subjects

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS
All the titles listed below can be obtained from good booksellers or from Cambridge University Press. For a complete series listing, visit www.cambridge.org/mathematics

129 J. Berstel, D. Perrin, and C. Reutenauer *Codes and Automata*
130 T. G. Faticoni *Modules over Endomorphism Rings*
131 H. Morimoto *Stochastic Control and Mathematical Modeling*
132 G. Schmidt *Relational Mathematics*
133 P. Kornerup and D. W. Matula *Finite Precision Number Systems and Arithmetic*
134 Y. Crama and P. L. Hammer (eds.) *Boolean Models and Methods in Mathematics, Computer Science, and Engineering*
135 V. Berthé and M. Rigo (eds.) *Combinatorics, Automata, and Number Theory*
136 A. Kristály, V. D. Rădulescu, and C. Varga *Variational Principles in Mathematical Physics, Geometry, and Economics*
137 J. Berstel and C. Reutenauer *Noncommutative Rational Series with Applications*
138 B. Courcelle and J. Engelfriet *Graph Structure and Monadic Second-Order Logic*
139 M. Fiedler *Matrices and Graphs in Geometry*
140 N. Vakil *Real Analysis through Modern Infinitesimals*
141 R. B. Paris *Hadamard Expansions and Hyperasymptotic Evaluation*
142 Y. Crama and P. L. Hammer *Boolean Functions*
143 A. Arapostathis, V. S. Borkar, and M. K. Ghosh *Ergodic Control of Diffusion Processes*
144 N. Caspard, B. Leclerc, and B. Monjardet *Finite Ordered Sets*
145 D. Z. Arov and H. Dym *Bitangential Direct and Inverse Problems for Systems of Integral and Differential Equations*
146 G. Dassios *Ellipsoidal Harmonics*
147 L. W. Beineke and R. J. Wilson (eds.) with O. R. Oellermann *Topics in Structural Graph Theory*
148 L. Berlyand, A. G. Kolpakov, and A. Novikov *Introduction to the Network Approximation Method for Materials Modeling*
149 M. Baake and U. Grimm *Aperiodic Order I: A Mathematical Invitation*
150 J. Borwein et al. *Lattice Sums Then and Now*
151 R. Schneider *Convex Bodies: The Brunn-Minkowski Theory (Second Edition)*
152 G. Da Prato and J. Zabczyk *Stochastic Equations in Infinite Dimensions (Second Edition)*
153 D. Hofmann, G. J. Seal, and W. Tholen (eds.) *Monoidal Topology*
154 M. Cabrera García and Á. Rodríguez Palacios *Non-Associative Normed Algebras I: The Vidav-Palmer and Gelfand-Naimark Theorems*
155 C. F. Dunkl and Y. Xu *Orthogonal Polynomials of Several Variables (Second Edition)*
156 L. W. Beineke and R. J. Wilson (eds.) with B. Toft *Topics in Chromatic Graph Theory*
157 T. Mora *Solving Polynomial Equation Systems III: Algebraic Solving*
158 T. Mora *Solving Polynomial Equation Systems IV: Buchberger Theory and Beyond*
159 V. Berthé and M. Rigo (eds.) *Combinatorics, Words, and Symbolic Dynamics*
160 B. Rubin *Introduction to Radon Transforms: With Elements of Fractional Calculus and Harmonic Analysis*
161 M. Ghergu and S. D. Taliaferro *Isolated Singularities in Partial Differential Inequalities*
162 G. Molica Bisci, V. D. Radulescu, and R. Servadei *Variational Methods for Nonlocal Fractional Problems*
163 S. Wagon *The Banach-Tarski Paradox (Second Edition)*
164 K. Broughan *Equivalents of the Riemann Hypothesis I: Arithmetic Equivalents*
165 K. Broughan *Equivalents of the Riemann Hypothesis II: Analytic Equivalents*
166 M. Baake and U. Grimm (eds.) Aperiodic Order II: Crystallography and Almost Periodicity
167 M. Cabrera García and Á. Rodríguez Palacios *Non-Associative Normed Algebras II: Representation Theory and the Zel'manov Approach*
168 A. Yu. Khrennikov, S. V. Kozyrev, and W. A. Zúñiga-Galindo *Ultrametric Pseudodifferential Equations and Applications*
169 S. R. Finch *Mathematical Constants II*
170 J. Krajíček *Proof Complexity*
171 D. Bulacu, S. Caenepeel, F. Panaite and F. Van Oystaeyen *Quasi-Hopf Algebras*
172 P. McMullen *Geometric Regular Polytopes*
173 M. Aguiar and S. Mahajan *Bimonoids for Hyperplane Arrangements*
174 M. Barski and J. Zabczyk *Mathematics of the Bond Market: A Lévy Processes Approach*
175 T. R. Bielecki, J. Jakubowski, and M. Niewęgłowski *Fundamentals of the Theory of Structured Dependence between Stochastic Processes*
176 A. A. Borovkov *Asymptotic Analysis of Random Walks: Light-Tailed Distributions*
177 Y.-K. Chan *Foundations of Constructive Probability Theory*

Foundations of Constructive Probability Theory

YUEN-KWOK CHAN
Citigroup

CAMBRIDGE UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India

79 Anson Road, #06–04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of
education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781108835435
DOI: 10.1017/9781108884013

First published 2021

A catalogue record for this publication is available from the British Library.

Library of Congress Cataloging-in-Publication Data
Names: Chan, Yuen-Kwok, author.
Title: Foundations of constructive probability theory / Yuen-Kwok Chan.
Description: Cambridge, UK ; New York, NY : Cambridge University Press, 2021. |
Series: Encyclopedia of mathematics and its applications |
Includes bibliographical references and index.
Identifiers: LCCN 2020046705 (print) | LCCN 2020046706 (ebook) |
ISBN 9781108835435 (hardback) | ISBN 9781108884013 (epub)
Subjects: LCSH: Probabilities. | Stochastic processes. | Constructive mathematics.
Classification: LCC QA273 .C483 2021 (print) | LCC QA273 (ebook) |
DDC 519.2–dc23
LC record available at https://lccn.loc.gov/2020046705
LC ebook record available at https://lccn.loc.gov/2020046706

ISBN 978-1-108-83543-5 Hardback

Dedicated to the memory of my father, Tak-Sun Chan

Contents

Acknowledgments

Yuen-Kowk Chan is retired from Citigroup's Mortgage Analytics unit. All opinions expressed by the author are his own. The author is grateful to the late Professor E. Bishop for teaching him constructive mathematics, to the late Professors R. Getoor and R. Blumenthal for teaching him probability and for mentoring him, and to the late Professors R. Pyke and W. Birnbaum and the other statisticians in the Mathematics Department of the University of Washington, circa 1970s, for their moral support. The author is also thankful to the constructivists in the Mathematics Department of New Mexico State University, circa 1975, for hosting a sabbatical visit and for valuable discussions, especially to Professors F. Richman, D. Bridges, M. Mandelkern, W. Julian, and the late Professor R. Mines. Professors Melody Chan and Fritz Scholz provided incisive and valuable critiques of the introduction chapter of an early draft of this book. Professor Douglas Bridges gave many thoughtful comments of the draft. The author also wishes to thank Ms Jill Hobbs for her meticulous copyediting, and Ms Niranjana Harikrishnan for her aesthetically pleasing typography.

Nomenclature

$\equiv$ .... by definition equal to, 8
R .... set of real numbers, 8
d_{ecld} .... Euclidean metric, 8
$a \vee b$ .... $\max(a,b)$, 8
$a \wedge b$ .... $\min(a,b)$, 8
a_+ .... $\max(a,0)$, 8
a_- .... $\min(a,0)$, 8
$A \cup B$ .... union of sets A and B, 8
$A \cap B, AB$ .... intersection of sets A and B, 8
$[a]_1$ .... an integer $[a]_1 \in (a, a+2)$ for given $a \in R$, 9
$X|A$ .... restriction of function X on a set to a subset A, 9
$X' \circ X, X'(X)$ .... composite of functions X' and X, 10
$(X \leq a)$ .... $\{\omega \in domain(X) : X(\omega) \leq a\}$, 11
$X(\cdot, \omega'')$ .... function of first variable, given value of second variable for a function X of two variables, 12
$T^*(Y) \equiv T(\cdot, Y)$ .... dual function of Y relative to a certain mapping T, 12
(S,d) .... metric space, with metric d on set S, 12
$x \neq y$ .... $d(x,y) > 0$, where x, y are in some metric space, 13
J_c .... metric complement of subset J in a metric space, 12
$\bigotimes$ .... direct product of functions or sets, 14
$C_u(S,d), C_u(S)$ .... space of uniformly continuous real-valued functions on metric space (S,d), 14
$C_{ub}(S,d), C_{ub}(S)$ .... subspace of $C_u(S,d)$ whose members are bounded, 14
$C_0(S,d)$, $C_0(S)$ .... subspace of $C_u(S,d)$ whose members vanish at infinity, 15
$C(S,d), C(S)$ .... subspace of $C_u(S,d)$ whose members have bounded supports, 15
$\widehat{d}$ .... $1 \wedge d$, 15
O, o .... bounds for real-valued function, 16
$\square$ .... mark for end of proof or end of definition, 16
ξ .... binary approximation of a metric space, 20
$\|\xi\|$ .... modulus of local compactness corresponding to ξ, 20
$(\overline{S}, \overline{d})$ .... one-point compactification of (S,d), 34
Δ .... point at infinity, 34
$F|B$ .... $\{f|B : f \in F\}$, 35
$\int_{-\infty}^{+\infty} X(x)dF(x)$ .... Riemann–Stieljes integral, 46
1_A .... indicator of measurable set A, 67
A^c .... measure-theoretic complement of measurable set A, 67
$\diamond$ .... ordering between certain real numbers and functions, 73

(G, λ) .... profile system, 73
$(a,b) \ll \alpha$ .... the interval (a,b) is bounded in profile by α, 74
(Ω, L, E) .... probability space, 138
$\int E(d\omega)X(\omega)$ .... $E(X)$, 139
ρ_{Prob} .... the probability metric on r.v.'s, 145
$L(G)$ .... probability subspace generated by the family G of r.v.'s, 150
$\widehat{J}(S,d)$ .... set of distributions on complete metric space (S,d), 151
$\Rightarrow$ .... weak convergence of distributions, or convergence in distributions of r.r.v.'s, 155
$\rho_{Dist,\xi}$ .... metric on distributions on a locally compact metric space relative to binary approximation ξ, 156
F_X .... P.D.F. induced on R by an r.r.v. X, 165
$\Phi_{Sk,\xi}$ .... Skorokhod representation of distributions on (S,d), determined by ξ, 170
$L_{|L'}$ .... subspace of conditionally integrable r.r.v.'s given the subspace L', 184
$L_{|G}$ .... subspace of conditionally integrable r.r.v.'s, given $L(G)$, 184
E_A .... conditional expectation given an event A with positive probability, 184
$\varphi_{\overline{\mu},\overline{\sigma}}$ .... multivariate normal p.d.f., 192
$\Phi_{\overline{\mu},\overline{\sigma}}$ .... multivariate normal distribution, 193
$\varphi_{0,I}$ .... multivariate standard normal p.d.f., 193
$\Phi_{0,I}$ .... multivariate standard normal distribution, 193
Ψ .... tail of univariate standard normal distribution, 193
ψ_X .... characteristic function of r.v. X with values in R^n, 204
ψ_J .... characteristic function of distribution J on R^n, 204
$\widehat{g}$ .... Fourier transform of complex-valued function g on R^n, 204
$f \star g$ .... Convolution of complex-valued functions f and g on R^n, 204
ρ_{char} .... metric on characteristic functions on R^n, 210
$\widehat{R}(Q \times \Omega, S)$ .... set of r.f.'s with parameter set Q, state space (S,d), and sample space (Ω, L, E), 227
$X|K$ .... restriction of $X \in \widehat{R}(Q \times \Omega, S)$ to parameter subset $K \subset Q$, 227
$\delta_{Cp,K}$ .... modulus of continuity in probability of $X|K$, 228
$\delta_{cau,K}$ .... modulus of continuity a.u. of $X|K$, 228
$\delta_{auc,K}$ .... modulus of a.u. continuity of $X|K$, 228
$\widehat{F}(Q,S)$ .... set of consistent families of f.j.d.'s with parameter set Q and state space S, 232
$\widehat{\rho}_{Marg,\xi,Q}$ .... marginal metric for the set $\widehat{F}(Q,S)$ relative to the binary approximation ξ, 237
$\widehat{F}_{Cp}(Q,S)$ .... subset of $\widehat{F}(Q,S)$ whose members are continuous in probability, 238
$\widehat{\rho}_{Cp,\xi,Q,Q(\infty)}$ .... metric on $\widehat{F}_{Cp}(Q,S)$ relative to dense subset $Q_\infty$of parameter metric space Q, 240
$\widehat{\rho}_{Prob,Q}$ .... probability metric on $\widehat{R}(Q \times \Omega, S)$, 265
$\rho_{Sup,Prob}$ .... metric on $\widehat{F}_{Cp}(Q,S)$, 277
$\widehat{R}_{Cp}(Q \times \Omega, S)$ .... subset of $\widehat{R}(Q \times \Omega, S)$ whose members are continuous in probability, 228
$\widehat{R}_{Meas}(Q \times \Omega, S)$ .... subset of $\widehat{R}(Q \times \Omega, S)$ whose members are measurable, 276
$\widehat{R}_{Meas,Cp}(Q \times \Omega, S)$ .... $\widehat{R}_{Meas}(Q \times \Omega, S) \cap \widehat{R}_{Cp}(Q \times \Omega, S)$, 276
$\mathcal{L}$ .... filtration in probability space (Ω, L, E), 300
$\mathcal{L}_{\mathcal{X}}$ .... natural filtration of a process X, 300
$\mathcal{L}^+$ .... right-limit extension of filtration $\mathcal{L}$, 301
$L^{(\tau)}$ .... probability subspace of observables at stopping time τ relative to filtration $\mathcal{L}$, 302
$\overline{\lambda}$ .... the special convex function on R, 316
$Q_m, \overline{Q}_m, \widetilde{Q}_m, Q_\infty, \overline{Q}_\infty$ .... certain subsets of dyadic rationals in $[0,\infty)$, 332
$(\widehat{C}[0,1], \rho_{\widehat{C}[0,1]})$ .... metric space of a.u. continuous processes on $[0,1]$, 334

Φ_{Lim}	extension by limit of a process with parameter set Q_∞ to parameter set $[0,1]$, 335
$\widehat{D}[0,1]$	set of all a.u. càdlàg processes on $[0,1]$, 406
δ_{aucl}	modulus of a.u. càdlàg, 406
$\widehat{D}_{\delta(aucl),\delta(cp)}[0,1]$	subset of $\widehat{D}[0,1]$ whose members have moduli δ_{Cp}, and δ_{aucl}, 406
$\rho_{\widehat{D}[0,1]}$	metric on $\widehat{D}[0,1]$, 409
$(\widehat{R}_{Dreg}(Q_\infty \times \Omega, S), \widehat{\rho}_{Prob,Q(\infty)})$	metric space of D-regular processes, 410
Φ_{rLim}	extension by right-limit of a process with parameter set Q_∞ to parameter set $[0,1]$, 418
β_{auB}	modulus of a.u. boundedness, 445
δ_{SRCp}	modulus of strong right continuity in probability, 445
$\overline{\tau}_{f,a,N}(X)$	certain first exit times by the process X, 488
$\mathbf{T}$	a Markov semigroup, 500
$\delta_{\mathbf{T}}$	a modulus of strong continuity of $\mathbf{T}$, 500
$\alpha_{\mathbf{T}}$	a modulus of smoothness of $\mathbf{T}$, 500
$F^{*,\mathbf{T}}_{r(1),\cdots,r(m)}$	a finite joint transition distribution generated by $\mathbf{T}$, 502
$(\mathscr{T}, \rho_{\mathscr{T}})$	metric space of Markov semigroups. 525
$\mathbf{V}$	a Feller semigroup, 538
$\delta_{\mathbf{V}}$	a modulus of strong continuity of $\mathbf{V}$, 538
$\alpha_{\mathbf{V}}$	a modulus of smoothness of $\mathbf{V}$, 538
$\kappa_{\mathbf{V}}$	a modulus of nonexplosion of $\mathbf{V}$, 538
$F^{*,\mathbf{V}}_{r(1),\cdots,r(m)}$	a finite joint transition distribution generated by $\mathbf{V}$, 539
$((S,d),(\Omega,L,E),\{U^{x,\mathbf{V}} : x \in S\})$	Feller process, 543
$((R^m,d^m),(\Omega,L,E),\{B^x : x \in R^m\})$	Brownian motion as a Feller process, 568