

Foundations of Constructive Probability Theory

Using Bishop's work on constructive analysis as a framework, this monograph gives a systematic, detailed, and general constructive theory of probability theory and stochastic processes. It is the first extended account of this theory: Almost all of the constructive existence and continuity theorems that permeate the book are original. It also contains results and methods hitherto unknown in the constructive and nonconstructive settings. The text features logic only in the common sense and, beyond a certain mathematical maturity, requires no prior training in either constructive mathematics or probability theory. It will thus be accessible and of interest to both probabilists interested in the foundations of their specialty and constructive mathematicians who wish to see Bishop's theory applied to a particular field.

YUEN-KWOK CHAN completed a PhD in constructive mathematics with Errett Bishop before leaving academia for a career in private industry. He is now an independent researcher in probability and its applications.

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Dedicated to the memory of my father, Tak-Sun Chan

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Nomenclature

\equiv	by definition equal to, 8
R	set of real numbers, 8
$d_{ecl d}$	Euclidean metric, 8
$a \vee b$	$\max(a, b)$, 8
$a \wedge b$	$\min(a, b)$, 8
a_+	$\max(a, 0)$, 8
a_-	$\min(a, 0)$, 8
$A \cup B$	union of sets A and B , 8
$A \cap B, AB$	intersection of sets A and B , 8
$[a]_1$	an integer $[a]_1 \in (a, a + 2)$ for given $a \in R$, 9
$X A$	restriction of function X on a set to a subset A , 9
$X' \circ X, X'(X)$	composite of functions X' and X , 10
$(X \leq a)$	$\{\omega \in domain(X) : X(\omega) \leq a\}$, 11
$X(\cdot, \omega'')$	function of first variable, given value of second variable for a function X of two variables, 12
$T^*(Y) \equiv T(\cdot, Y)$	dual function of Y relative to a certain mapping T , 12
(S, d)	metric space, with metric d on set S , 12
$x \neq y$	$d(x, y) > 0$, where x, y are in some metric space, 13
J_c	metric complement of subset J in a metric space, 12
\otimes	direct product of functions or sets, 14
$C_u(S, d), C_u(S)$	space of uniformly continuous real-valued functions on metric space (S, d) , 14
$C_{ub}(S, d), C_{ub}(S)$	subspace of $C_u(S, d)$ whose members are bounded, 14
$C_0(S, d), C_0(S)$	subspace of $C_u(S, d)$ whose members vanish at infinity, 15
$C(S, d), C(S)$	subspace of $C_u(S, d)$ whose members have bounded supports, 15
\vec{d}	$1 \wedge d$, 15
O, o	bounds for real-valued function, 16
\square	mark for end of proof or end of definition, 16
ξ	binary approximation of a metric space, 20
$\ \xi\ $	modulus of local compactness corresponding to ξ , 20
(\vec{S}, \vec{d})	one-point compactification of (S, d) , 34
Δ	point at infinity, 34
$F B$	$\{f B : f \in F\}$, 35
$\int_{-\infty}^{+\infty} X(x)dF(x)$	Riemann–Stieljes integral, 46
1_A	indicator of measurable set A , 67
A^c	measure-theoretic complement of measurable set A , 67
\diamond	ordering between certain real numbers and functions, 73

(G, λ)	profile system, 73
$(a, b) \ll \alpha$	the interval (a, b) is bounded in profile by α , 74
(Ω, L, E)	probability space, 138
$\int E(d\omega)X(\omega)$	$E(X)$, 139
ρ_{Prob}	the probability metric on r.v.'s, 145
$L(G)$	probability subspace generated by the family G of r.v.'s, 150
$\hat{J}(S, d)$	set of distributions on complete metric space (S, d) , 151
\Rightarrow	weak convergence of distributions, or convergence in distributions of r.v.'s, 155
$\rho_{Dist, \xi}$	metric on distributions on a locally compact metric space relative to binary approximation ξ , 156
F_X	P.D.F. induced on \mathbb{R} by an r.v. X , 165
$\Phi_{Sk, \xi}$	Skorokhod representation of distributions on (S, d) , determined by ξ , 170
$L _{L'}$	subspace of conditionally integrable r.v.'s given the subspace L' , 184
$L _G$	subspace of conditionally integrable r.v.'s, given $L(G)$, 184
E_A	conditional expectation given an event A with positive probability, 184
$\varphi_{\bar{\mu}, \bar{\sigma}}$	multivariate normal p.d.f., 192
$\Phi_{\bar{\mu}, \bar{\sigma}}$	multivariate normal distribution, 193
$\varphi_{0, I}$	multivariate standard normal p.d.f., 193
$\Phi_{0, I}$	multivariate standard normal distribution, 193
Ψ	tail of univariate standard normal distribution, 193
ψ_X	characteristic function of r.v. X with values in \mathbb{R}^n , 204
ψ_J	characteristic function of distribution J on \mathbb{R}^n , 204
\hat{g}	Fourier transform of complex-valued function g on \mathbb{R}^n , 204
$f \star g$	Convolution of complex-valued functions f and g on \mathbb{R}^n , 204
ρ_{char}	metric on characteristic functions on \mathbb{R}^n , 210
$\hat{R}(Q \times \Omega, S)$	set of r.f.'s with parameter set Q , state space (S, d) , and sample space (Ω, L, E) , 227
$X K$	restriction of $X \in \hat{R}(Q \times \Omega, S)$ to parameter subset $K \subset Q$, 227
$\delta_{Cp, K}$	modulus of continuity in probability of $X K$, 228
$\delta_{cau, K}$	modulus of continuity a.u. of $X K$, 228
$\delta_{auc, K}$	modulus of a.u. continuity of $X K$, 228
$\hat{F}(Q, S)$	set of consistent families of f.j.d.'s with parameter set Q and state space S , 232
$\hat{\rho}_{Marg, \xi, Q}$	marginal metric for the set $\hat{F}(Q, S)$ relative to the binary approximation ξ , 237
$\hat{F}_{Cp}(Q, S)$	subset of $\hat{F}(Q, S)$ whose members are continuous in probability, 238
$\hat{\rho}_{Cp, \xi, Q, Q(\infty)}$	metric on $\hat{F}_{Cp}(Q, S)$ relative to dense subset Q_∞ of parameter metric space Q , 240
$\hat{\rho}_{Prob, Q}$	probability metric on $\hat{R}(Q \times \Omega, S)$, 265
$\rho_{Sup, Prob}$	metric on $\hat{F}_{Cp}(Q, S)$, 277
$\hat{R}_{Cp}(Q \times \Omega, S)$	subset of $\hat{R}(Q \times \Omega, S)$ whose members are continuous in probability, 228
$\hat{R}_{Meas}(Q \times \Omega, S)$	subset of $\hat{R}(Q \times \Omega, S)$ whose members are measurable, 276
$\hat{R}_{Meas, Cp}(Q \times \Omega, S)$	$\hat{R}_{Meas}(Q \times \Omega, S) \cap \hat{R}_{Cp}(Q \times \Omega, S)$, 276
\mathcal{L}	filtration in probability space (Ω, L, E) , 300
\mathcal{L}_X	natural filtration of a process X , 300
\mathcal{L}^+	right-limit extension of filtration \mathcal{L} , 301
$L^{(\tau)}$	probability subspace of observables at stopping time τ relative to filtration \mathcal{L} , 302
$\bar{\lambda}$	the special convex function on \mathbb{R} , 316
$Q_m, \bar{Q}_m, \bar{Q}_m, Q_\infty, \bar{Q}_\infty$	certain subsets of dyadic rationals in $[0, \infty)$, 332
$(\hat{C}[0, 1], \rho_{\hat{C}[0, 1]})$	metric space of a.u. continuous processes on $[0, 1]$, 334

Φ_{Lim}	extension by limit of a process with parameter set Q_∞ to parameter set $[0, 1]$, 335
$\widehat{D}[0, 1]$	set of all a.u. càdlàg processes on $[0, 1]$, 406
δ_{aucl}	modulus of a.u. càdlàg, 406
$\widehat{D}_{\delta(aucl), \delta(cp)}[0, 1]$	subset of $\widehat{D}[0, 1]$ whose members have moduli δ_{cp} , and δ_{aucl} , 406
$\rho_{\widehat{D}[0, 1]}$	metric on $\widehat{D}[0, 1]$, 409
$(\widehat{R}_{Dreg}(Q_\infty \times \Omega, S), \widehat{\rho}_{Prob, Q(\infty)})$	metric space of D -regular processes, 410
Φ_{rLim}	extension by right-limit of a process with parameter set Q_∞ to parameter set $[0, 1]$, 418
β_{auB}	modulus of a.u. boundedness, 445
δ_{SRCp}	modulus of strong right continuity in probability, 445
$\overline{\tau}_{f, a, N}(X)$	certain first exit times by the process X , 488
\mathbf{T}	a Markov semigroup, 500
$\delta_{\mathbf{T}}$	a modulus of strong continuity of \mathbf{T} , 500
$\alpha_{\mathbf{T}}$	a modulus of smoothness of \mathbf{T} , 500
$F_{r(1), \dots, r(m)}^{*, \mathbf{T}}$	a finite joint transition distribution generated by \mathbf{T} , 502
$(\mathcal{T}, \rho_{\mathcal{T}})$	metric space of Markov semigroups, 525
\mathbf{V}	a Feller semigroup, 538
$\delta_{\mathbf{V}}$	a modulus of strong continuity of \mathbf{V} , 538
$\alpha_{\mathbf{V}}$	a modulus of smoothness of \mathbf{V} , 538
$\kappa_{\mathbf{V}}$	a modulus of nonexplosion of \mathbf{V} , 538
$F_{r(1), \dots, r(m)}^{*, \mathbf{V}}$	a finite joint transition distribution generated by \mathbf{V} , 539
$((S, d), (\Omega, L, E), \{U^x, \mathbf{V} : x \in S\})$	Feller process, 543
$((R^m, d^m), (\Omega, L, E), \{B^x : x \in R^m\})$	Brownian motion as a Feller process, 568

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