Cambridge University Press
978-1-108-83543-5 — Foundations of Constructive Probability Theory
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Excerpt
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Part I

Introduction and Preliminaries

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Introduction

The present work on probability theory is an outgrowth of the constructive analysis in [Bishop 1967] and [Bishop and Bridges 1985].

Perhaps the simplest explanation of constructive mathematics is by way of focusing on the following two commonly used theorems. The first, the *principle of finite search*, states that, given a finite sequence of 0-or-1 integers, either all members of the sequence are equal to 0, or there exists a member that is equal to 1. We use this theorem without hesitation because, given the finite sequence, a finite search would determine the result.

The second theorem, which we may call the *principle of infinite search*, states that, given an infinite sequence of 0-or-1 integers, either all members of the sequence are equal to 0, or there exists a member that is equal to 1. The name "infinite search" is perhaps unfair, but it brings into sharp focus the point that the computational meaning of this theorem is not clear. The theorem is tantamount to an infinite loop in computer programming without any assurance of termination.

Most mathematicians acknowledge the important distinction between the two theorems but regard the principle of infinite search as an expedient tool to prove theorems, with the belief that theorems so proved can then be specialized to constructive theorems, when necessary. Contrary to this belief, many classical theorems proved directly or indirectly via the principle of infinite search are actually equivalent to the latter: as such, they do not have a constructive proof. Oftentimes, not even the numerical meaning of the theorems in question is clear.

We believe that, for the constructive formulations and proofs of even the most abstract theorems, the easiest way is to employ a disciplined and systematic approach, by using only finite searches and by quantifying mathematical objects and theorems at each and every step, with natural numbers as a starting point. The references cited earlier show that this approach is not only possible but also fruitful.

It should be emphasized that we do not claim that theorems whose proofs require the principle of infinite search are untrue or incorrect. They are certainly correct and consistent derivations from commonly accepted axioms. There is indeed no reason why we cannot discuss such classical theorems alongside their constructive counterparts. The term "nonconstructive mathematics" is not meant to be pejorative. We will use, in its place, the more positive term "classical mathematics."

Moreover, it is a myth that constructivists use a different system of logic. The only logic we use is commonsense logic; no formal language is needed. The present author considers himself a mathematician who is neither interested in nor equipped to comment on the formalization of mathematics, whether classical or constructive.

Since a constructively valid argument is also correct from the classical viewpoint, a reader of the classical persuasion should have no difficulties understanding our proofs. Proofs using only finite searches are surely agreeable to any reader who is accustomed to infinite searches.

Indeed, the author would consider the present book a success if the reader, but for this introduction and occasional remarks in the text, finishes reading without realizing that this is a constructive treatment. At the same time, we hope that a reader of the classical persuasion might consider the more disciplined approach of constructive mathematics for his or her own research an invitation to a challenge.

Cheerfully, we hasten to add that we do not think that finite computations in constructive mathematics are the end. We would prefer a finite computation with n steps to one with $n!$ steps. We would be happy to see a systematic and general development of mathematics that is not only constructive but also computationally efficient. That admirable task will, however, be left to abler hands.

Probability theory, which is rooted in applications, can naturally be expected to be constructive. Indeed, the crowning achievements of probability theory – the laws of large numbers, the central limit theorems, the analysis of Brownian motion processes and their stochastic integrals, and the analysis of Levy processes, to name just a few – are exemplars of constructive mathematics. Kolmogorov, the grandfather of modern probability theory, actually took an interest in the formalization of general constructive mathematics.

Nevertheless, many a theorem in modern probability actually implies the principle of infinite search. The present work attempts a systematic constructive development. Each existence theorem will be a construction. The input data, the construction procedure, and the output objects are the essence and integral parts of the theorem. Incidentally, by inspecting each step in the procedure, we can routinely observe how the output varies with the input. Thus a continuity theorem in epsilon–delta terms routinely follows an existence theorem. For example, we will construct a Markov process from a given semigroup and prove that the resulting Markov process varies continuously with the semigroup.

The reader familiar with the probability literature will notice that our constructions resemble Kolmogorov's construction of the Brownian motion process, which is replete with Borel–Cantelli estimates and rates of convergence. This is in contrast to popular proofs of existence via Prokhorov's theorem. The reader can regard Part III of this book, Chapters 6–11, the part on stochastic processes, as an extension of Kolmogorov's constructive methods to stochastic processes: Daniell–Kolmogorov–Skorokhod construction of random fields, measurable random fields, a.u. continuous processes, a.u. càdlàg processes, martingales, strong Markov processes, and Feller processes, all with locally compact state spaces.

Such a systematic, constructive, and general treatment of stochastic processes, we believe, has not previously been attempted.

The purpose of this book is twofold. First, a student with a general mathematics background at the first-year graduate-school level can use it as an introduction to probability or to constructive mathematics. Second, an expert in probability can use it as a reference for further constructive development in his or her own research specialties.

Part II of this book, Chapters 3–5, is a repackaging and expansion of the measure theory in [Bishop and Bridges 1985]. It enables us to have a self-contained probability theory in terms familiar to probabilists.

For expositions of constructive mathematics, see the first chapters of the last cited reference. See also [Richman 1982] and [Stolzenberg 1970]. We give a synopsis in Chapter 2, along with basic notations and terminologies for later reference.