Complexity Science

Ecosystems, the human brain, ant colonies and economic networks are all complex systems displaying collective behaviour, or emergence, beyond the sum of their parts. Complexity science is the systematic investigation of these emergent phenomena, and stretches across disciplines, from physics and mathematics, to biological and social sciences. This introductory textbook provides detailed coverage of this rapidly growing field, accommodating readers from a variety of backgrounds and with varying levels of mathematical skill. Part I presents the underlying principles of complexity science, to ensure students have a solid understanding of the conceptual framework. Part II introduces the key mathematical tools central to complexity science, gradually developing the mathematical formalism, with more advanced material provided in boxes. A broad range of end-of-chapter problems and extended projects offer opportunities for homework assignments and student research projects, with solutions available to instructors online. Key terms are highlighted in bold and listed in a glossary for easy reference, while annotated reading lists offer the option for extended reading and research.

Henrik Jeldtoft Jensen is Professor of Mathematical Physics at Imperial College London, and leads the Centre for Complexity Science. He is a prominent expert in complexity science and is involved in a variety of high-profile research projects including the application of co-evolutionary dynamics to the modelling of socio-economical sustainability, finance, cultural evolution, innovation and cell diversity in cancer tumour growth, and has also worked with the Guildhall School of Music and Drama to identify differences in the neuronal response of audience and performers depending on the mode of performance. He has published two books on self-organised criticality and complex systems.

Complexity Science

The Study of Emergence

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To Vibeke, Barbara and Rebecca

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Acknowledgements

My perception of what complexity science is, and my understanding of how it is best pursued, have been developed over many years of fortunate exchanges of views, ideas and expertise through collaboration and dialogue with a wealth of people. Some scientists, some artists and musicians, and some with other backgrounds. It is not possible for me to pinpoint the exact effect of each conversation and each person, but I know that my thinking and knowledge have grown from this lush environment of inquisitive curiosity. Around the world I have been lucky to be with people sharing the curiosity to understand, for the sake of contributing to our comprehension of our surroundings. The kind of people convinced that investigating basic *why and how* questions can lead to multitudes of valuable applications of a nature that we were not able to imagine at the outset.

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Nomenclature

The following table contains, in alphabetical order, the Greek alphabet followed by the Roman, mathematical symbols, a brief description and the page of their first use.

Symbol	Description	Page
α	Phase lag	171
β	Inverse of the temperature times Boltzmann's constant	112
β	Parameter in the transition probability	314
β	Critical exponent for order parameter	126
γ	Drift velocity	285
γ_{Tot}	Total restriction ratio	261
γ_{mar}	Marginal restriction factor	267
γ_T	Product of reduction factors	268
γ_R	Relative reduction factor	269
$\gamma_{\neg}$	Complement restriction factor	267
δ	Size of change caused by mutation	32
δ	Asymmetry of random walker	284
$\delta_{i,j}$	Kronecker delta function	96
$\delta(x-x_0)$	Dirac delta function	287
$\Delta h(x,y)$	Change in function *h*	115
$\Delta(AB)$	Group entropic complexity measure	273
Δu_{ag}	Change in agent utility	330
ΔU_{BA}	Change in potential function	314
ϵ	Coupling strength between Kuramoto rotors	164
η	Coefficient of restoring force	307
θ_i	Angle of rotor or arrow	143
λ_q	Eigenvalue number *q*	185
μ	Carrying capacity-like coefficient	338
μ	Average number of branches	93
ν	Lack of conservation parameter	361
ξ	Correlation length	130
π_Q	Cluster size probability	198
π_{ij}	Probability walker moves from node *i* to node *j*	213
$\mathbf{\Pi}$	Transition matrix	213
$\boldsymbol{\rho}$	Vector of local densities	316
ρ_i	Component of vector of local densities	316
ρ_Q	Probability of size of cluster at the end of link	199
σ	Random velocity coefficient	307
σ_i	Component of occupancy vector	315
$\boldsymbol{\sigma}$	Occupancy vector	315
σ_X	Standard deviation	232
Σ	Structural entropy	270
τ	Time to extinction	102
τ	Correlation time	130
χ	Magnetic susceptibility	127
Υ	Helicity modulus	147

Symbol	Description	Page
$\Phi(x, y)$	Composition function	251
$\psi_k(t)$	Deviation from the average of rotor k	165
$\chi(t)$	Stochastic force	297
$\bar{\omega}$	Average rotor velocity	165
ω_k	Speed of rotor number k	164
Ω	Number of possible events or states	113
Ω	Organisation entropy	270
$\Omega(E, V)$	Total number of microstates possible under these constraints	111
$\vec{\nabla} f$	Gradient of function f	114
a	Random walker step size	280
a_{ij}	Element of adjacency matrix	179
$\boldsymbol{A}$	Adjacency matrix	179
B	Applied magnetic field	120
$\tilde{B}_i$	Effective field	122
C	Heat capacity	127
C	Connectance	179
$C(r, t)$	Correlation function	129
C_q	Network cluster number q	190
C_{XY}	Correlation coefficient	232
$C_{LZ}(s)$	Lempel–Ziv measure	258
D	Diffusion constant	285
$DTE_{X_i \to X_j}$	Direct transfer entropy	243
E	Energy	111
$E_{\mathbf{B}}$	Energy of bath	111
E_{Tot}	Energy of bath and system	111
E_{S}	Energy of system	111
$E(\mathbf{r})$	Dynamical variable of the Zhang model	359
$\langle E \rangle$	Average energy	117
$F(\boldsymbol{\rho})$	Stochastic process free energy	317
$F(x)$	Accumulated probability density	371
$\hat{f}(k)$	Fourier transform of function $f(x)$	286
$g(x, t)$	Source term	297
$g_0(s)$	Generator function for the degree distribution	197
$g_1(s)$	Generator for the excess degree distribution	197
$g_X(s)$	Generator function for the stochastic variable X	94
$g(\omega)$	Density of Kuramoto rotors	165
$G(t)$	Group generator function	252
$G(\boldsymbol{\rho})$	Stochastic process potential	331
H	Hurst exponent	303
$H(X)$	Shannon entropy	237
$H(X\|Y)$	Conditioned entropy	239
$H[S_1, \ldots, S_N]$	Hamiltonian of the Ising spins S_i	120
$H(\mathbf{S}, t)$	Offspring probability weight function	338
$H_{\neg 1}$	Entropy conditioned on complement to $\neg 1$	267
$I(X; Y)$	Mutual information	233
J	Strength of interaction in the Ising model	120
J_{ij}	Strength of interaction in the Ising spin glass	375
$J(x, t)$	Current	293
$J(\mathbf{S}, \mathbf{S}')$	Interaction strength between type $\mathbf{S}$ and type $\mathbf{S}'$	338
k	Degree of node	179
k_i	Degree of node i	180
k_B	Boltzmann's constant	9
K	Kuramoto order parameter	165
L_{C_q}	Number of links in cluster C_q	190

Symbol	Description	Page
L	Number of links in network	179
$L(x)$	Lambert function	254
$n(\psi)$	Rotor density	167
$n_{as}(\psi)$	Density of non-synchronised rotors	167
$n_s(\psi)$	Density of synchronised rotors	167
$n(x,t)$	Number of agents of type x at time t	309
N	Number of nodes in network	179
N	Number of particles	9
m	Magnetisation per spin	120
$\langle m \rangle$	Thermal average of magnetisation per spin	120
$\frac{\partial f}{\partial x}$	Partial derivative of function f with respect to x	114
p	Pressure	9
p_+	Right step probability	280
p_-	Left step probability	280
p_i	Probability of event i	113
$p(s)$	Probability that the system is in a particular state s	111
p_k	Branching probability	93
p_{mut}	Mutation probability	309
$P_{\mathrm{deg}}(k)$	Degree distribution	179
$p_{\mathrm{off}}(\mathbf{S},t)$	Offspring probability	338
p_{kill}	Killing probability	340
$P_{\mathrm{mut}}^{(0)}$	Probability that no ‘genes’ mutate	342
$P_{1\mathrm{pass}}(t)$	First passage time probability	293
$P_{1\mathrm{ret}}(t)$	First return time probability	296
$P_i(t)$	Probability system is in state i at time t	282
$P(x,t)$	Probability system is at x at time t	281
$P(k)$	Probability of event k	53
$P_X(n)$	Probability that $X = n$	94
$P_{X_i,X_j}(x_i,x_j)$	Joint probability	231
$P_{\mathbf{X}}(x_1,x_2,\ldots,x_N)$	Simultaneous probability	231
$P_{X\mid Y}(x,y)$	Conditioned probability	238
$P(Q\|k)$	Probability of size of cluster of k neighbours	199
q	Number of neighbours	122
q_i	Number of agents on node i	213
$\boldsymbol{Q}$	Vector of occupancies	213
Q	Number of configurations each component can occupy	261
$Q[C]$	Modularity of partitioning C	191
s_k	Step size	302
$s(\rho_q)$	Entropy of state ρ_q	316
$s(l_1,l_2)$	Sequence factor	257
S	Thermodynamic entropy	112
S	Strength of giant cluster	201
S	Avalanche size	361
$\mathbf{S}_\alpha$	Agent label	338
S_α	Rényi entropy	252
$S(N)$	Sequence	257
$S[p]$	Group entropy	252
$S[\mathbf{p}]$	Shannon entropy	113
S_q	Tsallis’s q entropy	250
$\langle S \rangle$	Thermodynamic average of an arbitrary spin S	121
$\langle S_i \rangle$	Thermodynamic average of a specific spin S_i	121
$\boldsymbol{S}_i$	Ising variable at position i equal to $+1$ or -1	48
t_i	Record time	372
T	Temperature	9

Symbol	Description	Page
T	Avalanche duration	361
T_{BA}	Transition probabilities	314
T_{ij}	Transition probabilities	282
$TE_{Y \to X}$	Transfer entropy	236
Tr $\boldsymbol{A}$	Trace of matrix $\boldsymbol{A}$	135
$u(\rho_q)$	Coarse-grained potential function	316
u_{ag}	Agent utility	330
U_{gl}	Global utility	329
$U(A)$	Potential function	314
$\boldsymbol{v}_q$	Eigenvector number q	185
V	Volume	9
W_{Tot}	Effective total number of allowed configurations	261
W_i	Effective number of allowed configurations of i	267
$W_{\neg i}$	Effective number configurations in the complement of i	267
x	Agent position along label axis	32
x_d	Position at which agent dies	32
x_0	Initial position of population of agents	32
x_{off}	Position of offspring	31
x_n	Probability that branching process stops at generation n	98
X_i	The ith component of a time series vector	230
$\mathbf{X}(t)$	Time series vector	230
Y_n	Total number of decedents up to generation n	97
Y_∞	Total size of the progeny	97
$z(\mathbf{r})$	Sandpile dynamical variable	358
Z	Partition function	112
Z_n	Size of generation n	96
$Z_n^{(k)}$	Number of nodes in generation n originating from node k in the first generation	96

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Preface

Many good books on complexity science currently exist. Some discuss complexity science from the perspective of a specific methodology such as network theory, analysis of power laws or use of agent-based simulations, while others discuss real systems which are considered to be complex, such as finance, sociology or ecology. References will be given where relevant to the context throughout the following pages.

This textbook is different in its aim and format from existing books. The book will present complexity science as a science in its own right, which focuses on the systematic study of emergent phenomena. Figure 1 is included here to indicate from the onset what our focus will be. The figure is best read with a concrete example of the components,

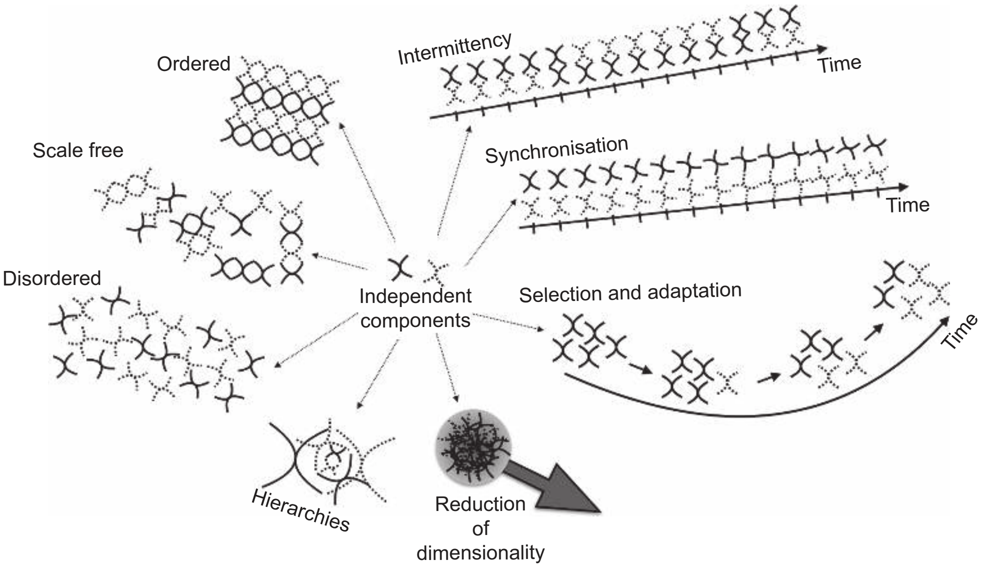

Figure 1 Emerging structure. Two components, agents, are shown in the centre. We now contemplate what will happen when collections of such components are made to interact and form collective structures. Surrounding the agents are sketches of structures and patterns that may emerge in time and space at the aggregate level. Types of static structures are to the left. Order and disordering tendencies may compete and produce a richness of properties with no equivalence amongst the individual agents. The dynamics of the interacting components can produce a wealth of different collective dynamical modes. For example, intermittent rearrangement, synchronisation and selection leading to adaptation. Or the interaction between components may produce a rigidity that allows the description of the time dependence in terms of just a few parameters, allowing us not to specify the dynamics of the individual components. This corresponds to a reduction of dimensionality.

or agents, in mind. They might, for example, be molecules forming rigid materials, neurones exhibiting synchronous firing patterns, biological cells building up hierarchies of structure, social agents segregating, etc. We will return to this figure in much more detail in Chap. 3; for now we simply suggest that the reader takes a careful look while thinking of possible manifestations of the examples sketched in this figure to obtain a feeling for what we will mean by the term ‘emergence’.

We will not try to define a complex system, because systems in general cannot be divided into two exclusive classes: the complex and the not complex. The situation is the same as if we try to define a physical system or a biological system. Most systems will have both physical and biological aspects. Think of a bird flying through the air. Its motion is subject to the physical laws of aerodynamics but its metabolism and physiology are part of biology. This is the reason why we focus on the concepts and methodology of complexity science and take the viewpoint that its subject matter is emergence rather than specific types of systems. We will, however, as we go along mention many examples of applications of complexity science to real systems.

This textbook is intended for students, researchers and others with a wide variety of backgrounds. The book is separated into two parts. The first part is dedicated to a non-mathematical exposition of basic ideas and concepts used in complexity science. The second part develops a broad range of mathematical tools often used in complexity science. The hope is that the mathematics is introduced at a level that will be manageable for readers with just a basic high school background in calculus, vectors and matrices and probability theory. The presentation tries in a gentle way to gradually introduce the mathematical formalism and strives towards being self-contained, with some elaboration included as separate ‘technical boxes’. Concepts are boldfaced the first time they are introduced and a glossary presents brief definitions for rapid consultation.

The book’s structure is intended to make it useful for self-study as well as an accompanying text for a taught course. Each chapter is framed by a brief synopsis and summary. References to other textbooks and particularly relevant scientific papers are included throughout the text, and an annotated list of recommended Further Reading can be found at the end of each chapter. An online updated copy of the Further Readings can be found on the book’s webpage at Cambridge University Press. Exercises and projects are included at the end of each chapter. Some of these exercises are intended as brief discussions, perhaps with fellow students, to help digest the material. Others are more comprehensive and can, for example, be used as take-home assignments or even be developed into small research projects. In both cases, thinking through this material is expected to greatly help the reader to obtain a working knowledge of the field of complexity science.

The hope is that the reader will develop a clear understanding of the subject matter and the methodology of complexity science. The book will explain the focus of complexity science, its aim and how understanding complexity science can be helpful to people from various backgrounds and with different objectives. Think of the environment, our society, the economy, the mind or advanced IT systems. A biologist, a sociologist, a psychologist, an economist, a neuroscientist, a mathematician, a physicist

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and so forth may want to know about complexity science because of certain emergent aspects encountered while analysing a particular phenomenon, such as species diversity, social segregation, mental health, financial crashes, mental wellbeing or non-supervised machine learning.

Faced with the same kind of phenomena, the engineer, the decision maker, the politician and, for example, the journalist may want some familiarity with complexity science, its purpose and the kinds of analysis it can offer, for the simple reason that although complexity science cannot predict accurately the behaviour of large complex systems, it is able to help sort out what kind of behaviour can be expected.

The aspiration is that the book may serve as a guide to at least one possible path through the enormous and ever-growing terrain of concepts, methods, applications and literature of complexity science. Of course, the best way to read the book is to start at the beginning and work through to the end. But since for many it may be difficult to allocate the time needed for this, the book is written to make it possible to dive in and out of chapters and sections. This means that concepts will be considered multiple times from different perspectives throughout the book. Cross references should help to connect the discussion, but to make a non-sequential reading easier a certain amount of repetition occurs. The sequential reader can just make a nod of recognition and move on.

For easy reference, a list of mathematical notation is included at the front of the book and a glossary at the end of the book.