

The Art of Mathematics – Take Two

Lovers of mathematics, young and old, professional and amateur, will enjoy this book. It is mathematics with fun: a collection of attractive problems that will delight and test readers. Many of the problems are drawn from the large number that have entertained and challenged students, guests and colleagues over the years during afternoon tea. The problems have their roots in many areas of mathematics. They vary greatly in difficulty: some are very easy, but most are far from trivial, and quite a few rather hard. Many provide substantial and surprising results that form the tip of an iceberg, providing an introduction to an important topic.

To enjoy and appreciate the problems, readers should browse the book, choose one that looks particularly enticing, and think about it on and off for a while before resorting to the hint or the solution.

Follow threads for an enjoyable and enriching journey through mathematics.

BÉLA BOLLOBÁS has been a Fellow at Trinity College, Cambridge, for over fifty years, for decades as a Director of Studies in Mathematics, teaching the very best undergraduates in England, and is the Chair of Excellence in Combinatorics at the University of Memphis. He has had over seventy Ph.D. students. He is a Fellow of the Royal Society and a Member of the Academia Europaea, and a Foreign Member of the Hungarian Academy of Sciences and of the Polish Academy of Sciences. Among the awards he has received are a Senior Whitehead Prize (2007), a Bocskai Prize (2016), a Széchenyi Prize (2017) and an Honorary Doctorate from Adam Mickiewicz University, Poznań. This is his thirteenth book.

Learning is the greatest pleasure in life



The Art of Mathematics – Take Two
Tea Time in Cambridge

Béla Bollobás
*University of Cambridge and
University of Memphis*



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Contents

Preface	<i>page xi</i>
The Problems	1
The Hints	33
The Solutions	45
1. Real Sequences – An Interview Question	47
2. Vulgar Fractions – Sylvester’s Theorem	49
3. Rational and Irrational Sums	53
4. Ships in Fog	55
5. A Family of Intersections	56
6. The Basel Problem – Euler’s Solution	58
7. Reciprocals of Primes – Euler and Erdős	61
8. Reciprocals of Integers	65
9. Completing Matrices	66
10. Convex Polyhedra – Take One	68
11. Convex Polyhedra – Take Two	70
12. A Very Old Tripos Problem	72
13. Angle Bisectors – the Lehmus–Steiner Theorem	74
14. Langley’s Adventitious Angles	76
15. The Tantalus Problem – from <i>The Washington Post</i>	79
16. Pythagorean Triples	81
17. Fermat’s Theorem for Fourth Powers	84
18. Congruent Numbers – Fermat	86

19. A Rational Sum	89
20. A Quartic Equation	92
21. Regular Polygons	95
22. Flexible Polygons	99
23. Polygons of Maximal Area	100
24. Constructing $\sqrt[3]{2}$ – Philon of Byzantium	105
25. Circumscribed Quadrilaterals – Newton	108
26. Partitions of Integers	111
27. Parts Divisible by m and $2m$	114
28. Unequal vs Odd Partitions	115
29. Sparse Bases	117
30. Small Intersections – Sárközy and Szemerédi	119
31. The Diagonals of Zero–One Matrices	122
32. Tromino and Tetronimo Tilings	123
33. Tromino Tilings of Rectangles	126
34. Number of Matrices	128
35. Halving Circles	129
36. The Number of Halving Circles	131
37. A Basic Identity of Binomial Coefficients	134
38. Tepper’s Identity	136
39. Dixon’s Identity – Take One	138
40. Dixon’s Identity – Take Two	140
41. An Unusual Inequality	143
42. Hilbert’s Inequality	145
43. The Central Binomial Coefficient	147
44. Properties of the Central Binomial Coefficient	149
45. Products of Primes	151
46. The Erdős Proof of Bertrand’s Postulate	153
47. Powers of 2 and 3	155
48. Powers of 2 Just Less Than a Perfect Power	156
49. Powers of 2 Just Greater Than a Perfect Power	158
50. Powers of Primes Just Less Than a Perfect Power	159

Contents

vii

51. Banach’s Matchbox Problem	164
52. Cayley’s Problem	166
53. Min vs Max	168
54. Sums of Squares	170
55. The Monkey and the Coconuts	172
56. Complex Polynomials	174
57. Gambler’s Ruin	175
58. Bertrand’s Box Paradox	179
59. The Monty Hall Problem	181
60. Divisibility in a Sequence of Integers	185
61. Moving Sofa Problem	187
62. Minimum Least Common Multiple	191
63. Vieta Jumping	193
64. Infinite Primitive Sequences	195
65. Primitive Sequences with a Small Term	197
66. Hypertrees	199
67. Subtrees	200
68. All in a Row	201
69. An American Story	203
70. Six Equal Parts	205
71. Products of Real Polynomials	208
72. Sums of Squares	210
73. Diagrams of Partitions	212
74. Euler’s Pentagonal Number Theorem	214
75. Partitions – Maximum and Parity	218
76. Periodic Cellular Automata	220
77. Meeting Set Systems	223
78. Dense Sets of Reals – An Application of the Baire Category Theorem	225
79. Partitions of Boxes	227
80. Distinct Representatives	229
81. Decomposing a Complete Graph: The Graham–Pollak Theorem – Take One	230

82. Matrices and Decompositions: The Graham–Pollak Theorem – Take Two	232
83. Patterns and Decompositions: The Graham–Pollak Theorem – Take Three	234
84. Six Concurrent Lines	236
85. Short Words – First Cases	237
86. Short Words – The General Case	239
87. The Number of Divisors	241
88. Common Neighbours	243
89. Squares in Sums	244
90. Extension of Bessel’s Inequality – Bombieri and Selberg	245
91. Equitable Colourings	247
92. Scattered Discs	249
93. East Model	251
94. Perfect Triangles	254
95. A Triangle Inequality	256
96. An Inequality for Two Triangles	258
97. Random Intersections	260
98. Disjoint Squares	262
99. Increasing Subsequences – Erdős and Szekeres	264
100. A Permutation Game	266
101. Ants on a Rod	267
102. Two Cyclists and a Swallow	268
103. Almost Disjoint Subsets of Natural Numbers	270
104. Primitive Sequences	272
105. The Time of Infection on a Grid	274
106. Areas of Triangles: Routh’s Theorem	276
107. Lines and Vectors – Euler and Sylvester	282
108. Feuerbach’s Remarkable Circle	284
109. Euler’s Ratio–Product–Sum Theorem	286
110. Bachet’s Weight Problem	288
111. Perfect Partitions	291
112. Countably Many Players	294

Contents

ix

113. One Hundred Players	296
114. River Crossings: Alcuin of York – Take One	298
115. River Crossings: Alcuin of York – Take Two	301
116. Fibonacci and a Medieval Mathematics Tournament	303
117. Triangles and Quadrilaterals – Regiomontanus	305
118. The Cross-Ratios of Points and Lines	308
119. Hexagons in Circles: Pascal’s Hexagon Theorem – Take One	312
120. Hexagons in Circles: Pascal’s Theorem – Take Two	315
121. A Sequence in \mathbb{Z}_p	318
122. Elements of Prime Order	319
123. Flat Triangulations	320
124. Triangular Billiard Tables	322
125. Chords of an Ellipse: The Butterfly Theorem	324
126. Recurrence Relations for the Partition Function	326
127. The Growth of the Partition Function	328
128. Dense Orbits	332

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Preface

This collection of problems is a sequel to *The Art of Mathematics – Coffee Time in Memphis* (CTM). It is a playful tribute to four giants of mathematics and physics I was fortunate to know well: Paul Erdős, Paul Adrien Maurice Dirac, Israil Moiseevich Gelfand and John Edensor Littlewood. As in CTM, many of the problems in this volume are the kind they would have liked to think about. There are also echoes of the influence of my very early mathematical gurus, Baron Gábor Splényi and the geometer István Reiman.

I was horrified when I realized that as I have grown older, I have completely forgotten a number of the gems of classical elementary mathematics I had known so well in my early teens. It is for this reason that several of those gems have found their way into this collection. These ought to be well known by most people interested in mathematics.

This is not a volume for systematic study, but a book to enjoy. The problems have been selected for their beauty and the elegance of their solutions. Questions on the same topic are not collected into separate chapters, as I wanted to avoid the impression that this book can be used as a sound introduction to various topics. Rather, my hope is that the questions will whet the appetite of readers by giving them food for thought without much previous work.

Who are my intended readers? I have tried to make this volume appeal to people with vastly different backgrounds: students who enjoy doing mathematics, professional mathematicians looking for some relaxation, and also all who loved mathematics in their younger days and are still happy to think about it. Even more, I hope that just about everyone in academia will benefit from dipping into this volume.

Some of the problems are very easy, but others are likely to be pretty demanding even for excellent mathematicians. My hope is that a reader will be fascinated by a problem or two, and will be happy to turn them over in his

head whether progress is forthcoming or not. I have always found that it is most pleasant to have a problem on my mind that I cannot do. Most problems have ‘hints’ that should give some help to the reader without taking away the pleasure of finding a complete solution.

The selection of problems shows a bias towards Cambridge mathematicians and, within Cambridge, towards members of Trinity College. As I have been a Fellow of Trinity College for over fifty years, I hope that this bias can be forgiven. My great respect for Trinity College was instilled in me close to sixty years ago by the Trinity mathematicians Harold Davenport and J.E. Littlewood, and the physicist Paul Dirac, who was actually a Fellow of St John’s College.

My chief desire has been to make the book readable; in particular, in the proofs I have not aimed for brevity: I often remind the reader of the relevant definitions and facts, so that he does not have to rack his brain to continue the proof. For this reason mathematicians are likely to find that the proofs go too slowly, but less experienced readers might welcome proofs spelled out in full!

The structure of this book is identical with that of CTM: the first section has the Problems, the second the Hints, and the third the Solutions, i.e. the proofs of the appropriate assertions. Needless to say, the reader should try to solve a problem without reading its hint, and get that help only when he is in dire need of it.

Most of the solutions are followed by notes which tend to be longer than in CTM, as they contain remarks not only about the mathematics, but also about the mathematicians involved with the problems.

I would be disappointed to learn that some people may read one of the problems, ponder about it for a minute or two, and then go on to read its solution. That would be a complete misuse of this volume, like hammering in a nail with a screwdriver. If a problem is found to need more mathematical expertise than the reader has, I would recommend abandoning that problem until such time as he has acquired the relevant background: this volume has plenty of problems that need little mathematical sophistication.

There are many people who drew my attention to beautiful problems, and gave me the pleasure of discussing those problems with them. I have received especially much help from Paul Balister (Oxford) and Imre Leader (Cambridge): I owe them a great debt of gratitude. I am also grateful to Józsi Balogh (Urbana), Enrico Bombieri (IAS, Princeton), Tim Gowers (Cambridge), Andrew Granville (Montreal), Misi Hujter (Budapest), Rob Morris (IMPA, Rio de Janeiro), Julian Sahasrabudhe (Cambridge), Tadashi Tokieda (Stanford) and Mark Walters (London).

This book would not have been finished without the great help I have received

Preface

xiii

from my brilliant friend of many decades, David Tranah, the Editorial Director for Mathematics at Cambridge University Press – much more than I could reasonably have hoped for. From the other side of the pond, my excellent Editorial Assistant, Tricia Simmons, has also given me much help. I am deeply grateful to both of them.

I should like to thank my current research students, Vojtěch Dvořák, Peter van Hintum, Harry Metrebian, Adva Mond, Jan Petr, Julien Portier, Victor Souza and Marius Tiba, for reading parts of the manuscript, and saving me from numerous howlers, like proving the necessity of a condition twice and forgetting to prove its sufficiency. I am sure that many mistakes have remained, for which I apologize.

Finally, this volume would never have been completed without the help and understanding of my wife, Gabriella, who has also put artistic ‘Art’ into it.

Béla Bollobás
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