The Art of Mathematics – Take Two

Lovers of mathematics, young and old, professional and amateur, will enjoy this book. It is mathematics with fun: a collection of attractive problems that will delight and test readers. Many of the problems are drawn from the large number that have entertained and challenged students, guests and colleagues over the years during afternoon tea. The problems have their roots in many areas of mathematics. They vary greatly in difficulty: some are very easy, but most are far from trivial, and quite a few rather hard. Many provide substantial and surprising results that form the tip of an iceberg, providing an introduction to an important topic.

To enjoy and appreciate the problems, readers should browse the book, choose one that looks particularly enticing, and think about it on and off for a while before resorting to the hint or the solution.

Follow threads for an enjoyable and enriching journey through mathematics.

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Learning is the greatest pleasure in life
The Art of Mathematics – Take Two

*Tea Time in Cambridge*

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Contents

Preface

The Problems

The Hints

The Solutions

1. Real Sequences – An Interview Question
2. Vulgar Fractions – Sylvester’s Theorem
3. Rational and Irrational Sums
4. Ships in Fog
5. A Family of Intersections
6. The Basel Problem – Euler’s Solution
7. Reciprocals of Primes – Euler and Erdős
8. Reciprocals of Integers
9. Completing Matrices
10. Convex Polyhedra – Take One
11. Convex Polyhedra – Take Two
12. A Very Old Tripos Problem
13. Angle Bisectors – the Lehmus–Steiner Theorem
14. Langley’s Adventitious Angles
15. The Tantalus Problem – from The Washington Post
16. Pythagorean Triples
17. Fermat’s Theorem for Fourth Powers
18. Congruent Numbers – Fermat
### Contents

19. A Rational Sum 89
20. A Quartic Equation 92
21. Regular Polygons 95
22. Flexible Polygons 99
23. Polygons of Maximal Area 100
24. Constructing $\sqrt{2}$ – Philon of Byzantium 105
25. Circumscribed Quadrilaterals – Newton 108
26. Partitions of Integers 111
27. Parts Divisible by $m$ and $2m$ 114
28. Unequal vs Odd Partitions 115
29. Sparse Bases 117
30. Small Intersections – Sárközy and Szemerédi 119
31. The Diagonals of Zero–One Matrices 122
32. Tromino and Tetronimo Tilings 123
33. Tromino Tilings of Rectangles 126
34. Number of Matrices 128
35. Halving Circles 129
36. The Number of Halving Circles 131
37. A Basic Identity of Binomial Coefficients 134
38. Tepper’s Identity 136
39. Dixon’s Identity – Take One 138
40. Dixon’s Identity – Take Two 140
41. An Unusual Inequality 143
42. Hilbert’s Inequality 145
43. The Central Binomial Coefficient 147
44. Properties of the Central Binomial Coefficient 149
45. Products of Primes 151
46. The Erdős Proof of Bertrand’s Postulate 153
47. Powers of 2 and 3 155
48. Powers of 2 Just Less Than a Perfect Power 156
49. Powers of 2 Just Greater Than a Perfect Power 158
50. Powers of Primes Just Less Than a Perfect Power 159
Contents

51. Banach’s Matchbox Problem 164
52. Cayley’s Problem 166
53. Min vs Max 168
54. Sums of Squares 170
55. The Monkey and the Coconuts 172
56. Complex Polynomials 174
57. Gambler’s Ruin 175
58. Bertrand’s Box Paradox 179
59. The Monty Hall Problem 181
60. Divisibility in a Sequence of Integers 185
61. Moving Sofa Problem 187
62. Minimum Least Common Multiple 191
63. Vieta Jumping 193
64. Infinite Primitive Sequences 195
65. Primitive Sequences with a Small Term 197
66. Hypertrees 199
67. Subtrees 200
68. All in a Row 201
69. An American Story 203
70. Six Equal Parts 205
71. Products of Real Polynomials 208
72. Sums of Squares 210
73. Diagrams of Partitions 212
74. Euler’s Pentagonal Number Theorem 214
75. Partitions – Maximum and Parity 218
76. Periodic Cellular Automata 220
77. Meeting Set Systems 223
78. Dense Sets of Reals – An Application of the Baire Category Theorem 225
79. Partitions of Boxes 227
80. Distinct Representatives 229
81. Decomposing a Complete Graph: The Graham–Pollak Theorem – Take One 230
Contents

82. Matrices and Decompositions: The Graham–Pollak Theorem – Take Two 232
83. Patterns and Decompositions: The Graham–Pollak Theorem – Take Three 234
84. Six Concurrent Lines 236
85. Short Words – First Cases 237
86. Short Words – The General Case 239
87. The Number of Divisors 241
88. Common Neighbours 243
89. Squares in Sums 244
90. Extension of Bessel’s Inequality – Bombieri and Selberg 245
91. Equitable Colourings 247
92. Scattered Discs 249
93. East Model 251
94. Perfect Triangles 254
95. A Triangle Inequality 256
96. An Inequality for Two Triangles 258
97. Random Intersections 260
98. Disjoint Squares 262
99. Increasing Subsequences – Erdős and Szekeres 264
100. A Permutation Game 266
101. Ants on a Rod 267
102. Two Cyclists and a Swallow 268
103. Almost Disjoint Subsets of Natural Numbers 270
104. Primitive Sequences 272
105. The Time of Infection on a Grid 274
106. Areas of Triangles: Routh’s Theorem 276
107. Lines and Vectors – Euler and Sylvester 282
108. Feuerbach’s Remarkable Circle 284
109. Euler’s Ratio–Product–Sum Theorem 286
110. Bachet’s Weight Problem 288
111. Perfect Partitions 291
112. Countably Many Players 294
This collection of problems is a sequel to The Art of Mathematics – Coffee Time in Memphis (CTM). It is a playful tribute to four giants of mathematics and physics I was fortunate to know well: Paul Erdős, Paul Adrien Maurice Dirac, Israil Moiseevich Gelfand and John Edensor Littlewood. As in CTM, many of the problems in this volume are the kind they would have liked to think about. There are also echoes of the influence of my very early mathematical gurus, Baron Gábor Splényi and the geometer István Reiman.

I was horrified when I realized that as I have grown older, I have completely forgotten a number of the gems of classical elementary mathematics I had known so well in my early teens. It is for this reason that several of those gems have found their way into this collection. These ought to be well known by most people interested in mathematics.

This is not a volume for systematic study, but a book to enjoy. The problems have been selected for their beauty and the elegance of their solutions. Questions on the same topic are not collected into separate chapters, as I wanted to avoid the impression that this book can be used as a sound introduction to various topics. Rather, my hope is that the questions will whet the appetite of readers by giving them food for thought without much previous work.

Who are my intended readers? I have tried to make this volume appeal to people with vastly different backgrounds: students who enjoy doing mathematics, professional mathematicians looking for some relaxation, and also all who loved mathematics in their younger days and are still happy to think about it. Even more, I hope that just about everyone in academia will benefit from dipping into this volume.

Some of the problems are very easy, but others are likely to be pretty demanding even for excellent mathematicians. My hope is that a reader will be fascinated by a problem or two, and will be happy to turn them over in his
head whether progress is forthcoming or not. I have always found that it is most pleasant to have a problem on my mind that I cannot do. Most problems have ‘hints’ that should give some help to the reader without taking away the pleasure of finding a complete solution.

The selection of problems shows a bias towards Cambridge mathematicians and, within Cambridge, towards members of Trinity College. As I have been a Fellow of Trinity College for over fifty years, I hope that this bias can be forgiven. My great respect for Trinity College was instilled in me close to sixty years ago by the Trinity mathematicians Harold Davenport and J.E. Littlewood, and the physicist Paul Dirac, who was actually a Fellow of St John’s College.

My chief desire has been to make the book readable; in particular, in the proofs I have not aimed for brevity: I often remind the reader of the relevant definitions and facts, so that he does not have to rack his brain to continue the proof. For this reason mathematicians are likely to find that the proofs go too slowly, but less experienced readers might welcome proofs spelled out in full!

The structure of this book is identical with that of CTM: the first section has the Problems, the second the Hints, and the third the Solutions, i.e. the proofs of the appropriate assertions. Needless to say, the reader should try to solve a problem without reading its hint, and get that help only when he is in dire need of it.

Most of the solutions are followed by notes which tend to be longer than in CTM, as they contain remarks not only about the mathematics, but also about the mathematicians involved with the problems.

I would be disappointed to learn that some people may read one of the problems, ponder about it for a minute or two, and then go on to read its solution. That would be a complete misuse of this volume, like hammering in a nail with a screwdriver. If a problem is found to need more mathematical expertise than the reader has, I would recommend abandoning that problem until such time as he has acquired the relevant background: this volume has plenty of problems that need little mathematical sophistication.

There are many people who drew my attention to beautiful problems, and gave me the pleasure of discussing those problems with them. I have received especially much help from Paul Balister (Oxford) and Imre Leader (Cambridge): I owe them a great debt of gratitude. I am also grateful to Józsi Balogh (Urbana), Enrico Bombieri (IAS, Princeton), Tim Gowers (Cambridge), Andrew Granville (Montreal), Misi Hujter (Budapest), Rob Morris (IMPA, Rio de Janeiro), Julian Sahasrabudhe (Cambridge), Tadashi Tokieda (Stanford) and Mark Walters (London).

This book would not have been finished without the great help I have received...
Preface

from my brilliant friend of many decades, David Tranah, the Editorial Director for Mathematics at Cambridge University Press – much more than I could reasonably have hoped for. From the other side of the pond, my excellent Editorial Assistant, Tricia Simmons, has also given me much help. I am deeply grateful to both of them.

I should like to thank my current research students, Vojtěch Dvořák, Peter van Hintum, Harry Metrebian, Adva Mond, Jan Petr, Julien Portier, Victor Souza and Marius Tiba, for reading parts of the manuscript, and saving me from numerous howlers, like proving the necessity of a condition twice and forgetting to prove its sufficiency. I am sure that many mistakes have remained, for which I apologize.

Finally, this volume would never have been completed without the help and understanding of my wife, Gabriella, who has also put artistic ‘Art’ into it.

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