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The Problems

1. (**Real Sequences**) (i) At most how many real numbers can be chosen from the open interval $(0, 2n+1)$ if none is at distance less than 1 from an integer multiple of another? To spell it out, let $n \geq 1$ be a fixed natural number. Suppose that $0 < x_1 < \cdots < x_N < 2n+1$ are such that $|kx_i - x_j| \geq 1$ for all natural numbers i, j and k with $1 \leq i < j \leq N$. At most how large is N?

(ii) At most how many real numbers can be chosen from the open interval $(0, (3n+1)/2) = (0, 3n/2 + 1/2)$ if none is at distance less than 1 from an *odd* multiple of another?

2. (**Vulgar Fractions**) Show that every rational number r, $0 < r < 1$, is the sum of a finite number of reciprocals of distinct natural numbers. For example,

$$\frac{4699}{7320} = \frac{1}{2} + \frac{1}{8} + \frac{1}{60} + \frac{1}{3660}.$$

3. (**Rational and Irrational Sums**) Let $2 \leq n_1 < n_2 < \cdots$ be a sequence of positive integers such that

$$n_{i+1} \geq n_i(n_i - 1) + 1$$

for every $i \geq 1$, and set

$$r = \sum_{i=1}^{\infty} \frac{1}{n_i}.$$

Show that r is rational if and only if $n_{i+1} = n_i(n_i - 1) + 1$ for all but finitely many values of i.

4. (**Ships in Fog**) Five ships, A, B, C, D and E are sailing in a fog with constant and different speeds, and constant and different straight-line courses, with different directions. The seven pairs AB, AC, AD, BC, BD, CE and DE

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have each had near misses, call them '*collisions*'. Does it follow that, in addition, E collides with either A or B? Maybe both? And does C collide with D?

5. (**A Family of Intersections**) For $0 < p < 1$, a *p-random subset* $X = X_p$ of $[n] = \{1, 2, \ldots, n\}$ is obtained by taking n independent binomial random variables $\xi_1, \xi_2, \ldots, \xi_n$ with $\mathbb{P}(\xi_i = 1) = p = 1 - \mathbb{P}(\xi_i = 0)$, and setting $X_p = \{i : \xi_i = 1\}$. The probability measure $\mathbb{P}_p$ on $\mathcal{P}_n$, the set of all 2^n subsets of $[n]$, is given by

$$\mathbb{P}_p(A) = \mathbb{P}(X_p = A) = p^{|A|} (1-p)^{n-|A|},$$

so that the probability of a family $\mathcal{A} \subset \mathcal{P}_n$ is

$$\mathbb{P}_p(\mathcal{A}) = \sum_{A \in \mathcal{A}} \mathbb{P}_p(A) = \sum_{A \in \mathcal{A}} p^{|A|} (1-p)^{n-|A|}.$$

Let $\mathcal{A} \subset \mathcal{P}_n$ have p-probability r: $\mathbb{P}_p(\mathcal{A}) = r$, and define

$$\mathcal{J} = \mathcal{J}(\mathcal{A}) = \{A \cap B : A, B \in \mathcal{A}\}.$$

Show that

$$\mathbb{P}_{p^2}(\mathcal{J}) \geq r^2.$$

6. (**The Basel Problem**) Forget for a moment the mathematical rigour we have to have in our proofs, and give a beautiful solution of the famous 'Basel Problem': prove that

$$\sum_{k=1}^{\infty} 1/k^2 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \cdots = \pi^2/6.$$

7. (**Reciprocals of Primes**) Give three proofs of the theorem that the sum of reciprocals of the primes is divergent: $\sum_p 1/p = \infty$, where the summation is over the primes.

8. (**Reciprocals of Integers**) Let $1 < n_1 < n_2 < \cdots$ be a sequence of natural numbers such that $\sum_{i=1}^{\infty} 1/n_i < \infty$. Show that the set

$$M = M(n_1, n_2, \ldots) = \{n_1^{\alpha_1} \ldots n_k^{\alpha_k} : \alpha_i \geq 0\}$$

has zero density, i.e. if $\varepsilon > 0$ and n is large enough (depending on ε) then there are at most εn elements of M that are at most n.

9. (**Completing Matrices**) For $1 \leq k < n$, let $\mathcal{A}_{k,n}$ be the collection of $n \times n$ matrices with each entry zero or one, having precisely k ones in each row and each column. Show that for $1 \leq r < n$ an $r \times n$ matrix of zeros and ones has an extension to a matrix in $\mathcal{A}_{k,n}$ if and only if each row has precisely k ones, and in each column there are at least $k + r - n$ and at most k ones.

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10. (**Convex Polyhedra – Take One**) Is there a convex polyhedron which contains a point whose perpendicular projection on the plane of every face falls outside the face? And just fails to fall in the interior of the face?

11. (**Convex Polyhedra – Take Two**) Show that every 3-dimensional polyhedron with at least thirteen faces has a face meeting at least six other faces. Two faces are said to meet if they share a vertex or an edge.

12. (**A Very Old Tripos Problem**) Let p, q and r be complex numbers with $pq \neq r$. Transform the cubic $x^3 - px^2 + qx - r = 0$, where the roots are a, b, c, into one whose roots are $\frac{1}{a+b}, \frac{1}{a+c}, \frac{1}{b+c}$.

13. (**Angle Bisectors**) Show that if two (internal) angle bisectors of a triangle are equal then the angles themselves are also equal.

14. (**Chasing Angles – Take One**) Let ABC be an isosceles triangle with angle 20° at the apex A. Let D be a point on AB and E a point on AC such that $\angle BCD = 50°$ and $\angle CBE = 60°$, as in Figure 1. What is the angle $\angle BED$?

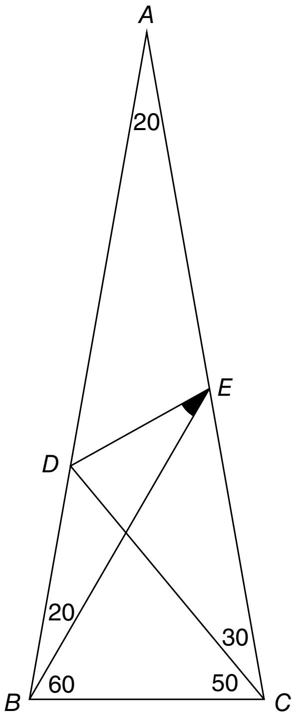

Figure 1 Adventitious angles

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15. (**Chasing Angles – Take Two**) Let ABC be an isosceles triangle with angle $20°$ at the apex A and so angles $80°$ at the base; furthermore, let D be a point on the side AB such that $\measuredangle CD = 10°$, and E on AC such that $\measuredangle BE = 20°$, as in Figure 2. Use entirely elementary methods, without any recourse to trigonometry, to determine the angle $\measuredangle DE$.

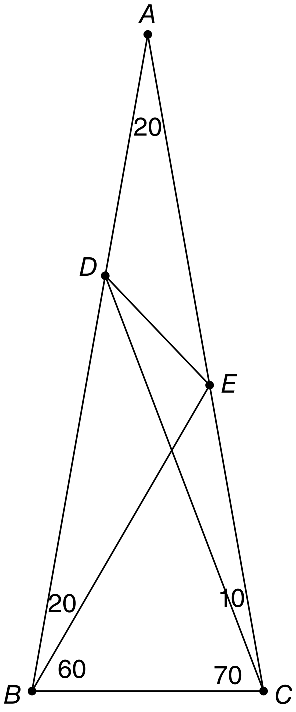

Figure 2 Information about our points

16. (**Pythagorean Triples**) We call a triple (a, b, c) of natural numbers a *Pythagorean triple* if $a^2+b^2 = c^2$. Also, a Pythagorean triple (a, b, c) is *primitive* or *relatively prime* if a, b and c do not have a common divisor (greater than 1). Clearly, every Pythagorean triple is a multiple of a primitive Pythagorean triple. Also, if (a, b, c) is a primitive Pythagorean triple then a and b have opposite parities since if both of them are odd then the sum of their squares is 2 modulo 4, so it cannot be a square, and if both of them are even then the sum of their squares is also even, so c also has to be even. Usually we take a to be odd and b even.

Show that (a, b, c) is a primitive Pythagorean triple with a odd and b even

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if and only if there are relatively prime numbers $u > v \geq 1$ of opposite parity such that $a = u^2 - v^2$, $b = 2uv$ and $c = u^2 + v^2$. Even more, give two proofs, one algebraic and the other geometric.

17. (**Fermat's Theorem for Fourth Powers**) Show that the equation $a^4 + b^4 = c^4$ has no solutions in natural numbers. Putting it slightly differently: if a and b are strictly positive integers then $a^4 + b^4$ cannot be a fourth power.

18. (**Congruent Numbers**) A natural number n is said to be *congruent* if there is a right-angled triangle with rational sides, whose area is n. For example, the right-angled triangle with sides 3, 4 and 5 tells us that 6 is congruent. Show that 1 is not a congruent number.

19. (**A Rational Sum**) Find a necessary and sufficient condition for a rational number $s > 1$ that ensures that $\sqrt{s+1} - \sqrt{s-1}$ is also rational, where $\sqrt{\cdot}$ denotes the positive square root.

20. (**A Quartic Equation**) Find a large family of integer solutions of

$$A^4 + B^4 = C^4 + D^4. \tag{1}$$

More precisely, look for fairly general polynomials A, B, C and D in $\mathbb{Z}[a, b]$ such that (1) holds. To this end, look for the solution in the form

$$A = ax + c, \qquad B = bx - d,$$
$$C = ax + d, \qquad D = bx + c$$

where a, b, c, d and x are rational numbers. Considering a, b, c and d constant, (1) holds if x satisfies a quartic whose first and last coefficients are 0. Show that with a suitable choice of a, b, c and d the coefficient of x^3 is also 0, and use this to find our polynomials.

21. (**Regular Polygons**) Show that, of all polygons of the same number of sides and equal perimeter length, the regular polygon has the greatest area.

22. (**Flexible Polygons**) Consider all polygons with given sides but one in a given cyclic order. Show that if the area of such a polygon is maximal then it may have a circle circumscribed about it, having the unknown side for a diameter of the circle.

23. (**Polygons of Maximal Area**) Show that the area of a polygon with given sides is not larger than the cyclic polygon with these sides, i.e. the one that may have a circle circumscribed about it. The reader is invited to find several solutions of this problem.

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24. (**Constructing $\sqrt[3]{2}$**) Let OS_1PS_2 be a $2m \times m$ rectangle with $OS_1 = PS_2$ of length $2m$ and $OS_2 = PS_1$ of length m. Let C be the circle through the vertices O, S_1, P and S_2, and let Q be the point of the PS_2 arc of C such that the line through P and Q meets the (extended) lines OS_1 and OS_2 in R_1 and R_2, and the segments PR_1 and QR_2 have the same length. Finally, let T_1 and T_2 be the projections of Q on the segments OR_1 and OR_2. Show that OT_1 has length $\sqrt[3]{2}m$.

25. (**Circumscribed Quadrilaterals**) Let $ABCD$ be a quadrilateral circumscribed about a circle with centre O. Let E and F be the midpoints of the

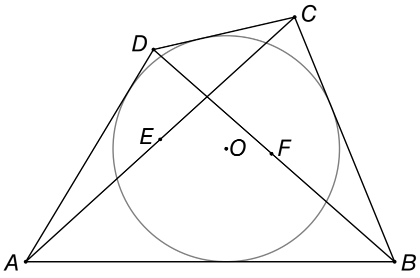

Figure 3 A quadrilateral circumscribed about a circle with centre O; the points E and F are the midpoints of the diagonals AC and BD.

diagonals AC and BD, as in Figure 3. Show that E, F and O are collinear.

26. (**Partitions of Integers**) A *partition* of an integer n is a sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ of positive integers $\lambda_1 \geq \cdots \geq \lambda_k \geq 1$ whose sum is n. This partition is also written as $\lambda_1 + \cdots + \lambda_k$. Each λ_i is a *summand* or *part*; the number of parts, k, is the *length* of the partition λ. As customary, we shall write $p(n)$ for the *partition function*, the number of partitions of $n \geq 1$. Note that 4 has five partitions: $4, 3+1, 2+2, 2+1+1, 1+1+1+1$, so $p(4) = 5$, and 5 has seven partitions: $5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1$, so $p(5) = 7$. Also, $p(0) = 1$: the only partition of 0 is the empty partition.

(i) Show that the formal power series $\sum_{n=0}^{\infty} p(n)x^n$, called the *generating function* of $p(n)$, is

$$(1 + x + x^2 + x^3 + \cdots)(1 + x^2 + x^4 + x^6 + \cdots)(1 + x^3 + x^6 + x^9 + \cdots)\cdots,$$

i.e.

$$\frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots .$$

(ii) Give three proofs of the assertion that the number of partitions of n

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without 1 as a part is $p(n) - p(n-1)$. For example, 5 has two partitions not containing 1, namely 5 and $3 + 2$, and $p(5) - p(4) = 7 - 5 = 2$.

27. (**Parts Divisible by** m **and** $2m$) Show that the number of partitions of n in which no multiple of m is repeated is equal to the number of partitions of n without a multiple of $2m$.

28. (**Unequal vs Odd Partitions**) (i) Show that the number of partitions of n into unequal parts is equal to the number of partitions into odd parts.

(ii) Let $m \geq 1$. Show that the number of partitions of n in which no part is repeated more than m times is equal to the number of partitions in which no part is a multiple of $m + 1$.

29. (**Sparse Bases**) A set S of natural numbers has density zero if $S(n)$ tends to zero as n tends to infinity, where $S(n)$ is the number of elements of S not greater than n.

Show that there is a set $S \subset \mathbb{N}$ of density zero such that every positive rational is the sum of a finite number of reciprocals of distinct terms of S.

30. (**Sets with Small Pairwise Intersections**) Let $A_1, \ldots, A_m \in [n]^{(r)}$, i.e. let $A_1, \ldots, A_m$ be r-subsets of $[n] = \{1, \ldots, n\}$. Show that if $|A_i \cap A_j| \leq s < r^2/n$ for all $1 \leq i < j \leq m$, then $m \leq n(r - s)/(r^2 - sn)$.

Show also that if r^2/n is an integer and $|A_i \cap A_j| < r^2/n$ for all $1 \leq i < j \leq m$, then $m \leq r - r^2/n + 1 \leq n/4 + 1$.

31. (**The Diagonals of Zero–One Matrices**) Given $n \geq 1$, let $\mathcal{A}_n$ be the set of all $n \times n$ matrices with each entry 0 or 1. For $A \in \mathcal{A}_n$, write $\mathcal{A}(A)$ for the set of matrices obtained from A by permuting its rows. Thus if no two rows of A are equal then $\mathcal{A}(A)$ consists of $n!$ matrices. Denote by $d(A)$ the number of different main diagonals of the matrices in $\mathcal{A}(A)$. Determine $d(n) = \max\{d(A) : A \in \mathcal{A}_n\}$.

Note that the total number of main diagonals (with entries 0 and 1) is 2^n, which is much smaller than $n!$, so there is no obvious reason why we could not obtain all 2^n diagonals. For example, if the three rows of our 3×3 matrix are 111, 101 and 001, then the diagonals are 101 (if 111 is kept as the first row), 111, 011 and 001 (for the other orders).

32. (**Tromino and Tetronimo Tilings**) An $m \times n$ *board* is an $m \times n$ rectangle made up of mn unit squares called cells; a *deficient* $m \times n$ *board* $m \times n$ board from which a cell has been removed. A *tromino* is the union of three cells sharing a vertex, and a T-*tetromino* is the union of four cells in the shape of a

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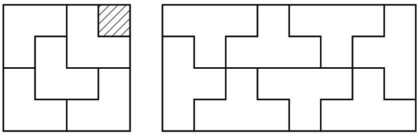

Figure 4 A tromino tiling of a deficient 4×4 board, and a T-tetromino tiling of a 4×8 board.

letter T. We are interested in tilings of a deficient $n \times n$ board by trominoes and an $m \times n$ board by T-tetrominoes, as in Figure 4.

(i) Show that if n is a power of 2 then every deficient $n \times n$ board can be tiled with trominoes.

(ii) Show that if an $m \times n$ rectangle can be tiled with T-tetrominoes then mn is divisible by 8.

33. (**Tromino Tilings of Rectangles**) For what values of m and n can an $m \times n$ rectangle be tiled with trominoes, as in Figure 5? [A tromino is a 2×2 square with one quarter cut off, as in Problem 32.]

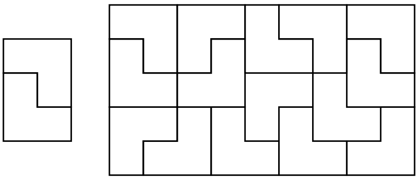

Figure 5 Tromino tilings of a 3×2 board and a 5×9 board.

34. (**Number of Matrices**) What is the number of $n \times n$ matrices with non-negative integer entries, in which every row and column has at most three non-zero entries, these non-zero entries are different, and their sum is 7? An example of such a matrix is

$$\begin{pmatrix} 0 & 1 & 0 & 2 & 4 \\ 5 & 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 & 3 \\ 0 & 6 & 0 & 1 & 0 \\ 2 & 0 & 1 & 4 & 0 \end{pmatrix}$$

35. (**Halving Circles**) Let S be a set of $2n + 1 \geq 5$ points in the plane in general position. In this context, being in ‘general position’ means that no three

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points are on a line and no four points are on a circle. We say that a circle C *halves* S if three points of S are on C, $n-1$ inside C and so $n-1$ outside C. Show that there are at least $n(2n+1)/3$ halving circles.

36. (**The Number of Halving Circles**) Continuing the previous problem, show that for every $n \geq 1$ there is a set of $2n+1$ points in general position in the plane with exactly n^2 halving circles.

37. (**A Basic Identity of Binomial Coefficients**) Let $f(X)$ be a polynomial of degree less than n. Show that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}f(k) = 0.$$

38. (**A Simple Sum?**) Put the sum

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}(x-i)^n$$

into a much simpler form.

39. (**Dixon's Identity – Take One**) We shall use the convention that $0! = 1$ and $1/k! = 0$ for $k < 0$. Let a, b, c be non-negative integers. Show that

$$\sum_k \frac{(-1)^k(a+b)!(b+c)!(c+a)!}{(a+k)!(a-k)!(b+k)!(b-k)!(c+k)!(c-k)!} = \frac{(a+b+c)!}{a!b!c!}.$$

On the left-hand side, the summation is over all integers k; equivalently, we may take the sum $\sum_{-d\leq k\leq d}$, where $d = \min\{a,b,c\}$.

40. (**Dixon's Identity – Take Two**) (i) Let m and n be non-negative integers, and write X for a variable. Prove the following identity of polynomials with real coefficients:

$$\sum_{k=0}^{2n}(-1)^k\binom{m+2n}{m+k}\binom{X}{k}\binom{X+m}{m+2n-k} = (-1)^n\binom{X}{n}\binom{X+m+n}{m+n}.$$

Here and elsewhere, for a polynomial $f(X)$ over the reals and a non-negative integer ℓ, we write

$$\binom{f(X)}{\ell} = f(X)(f(X)-1)(f(X)-2)\ldots(f(X)-\ell+1).$$

In particular, $\binom{f(X)}{0} = 1$, and if ℓ is a negative integer then $\binom{f(X)}{\ell} = 0$.
(ii) Deduce that if a, b and c are non-negative integers and, say, $b \leq a, c$, then

$$\sum_{k=-b}^{b}(-1)^k\binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+a}{c+k} = \frac{(a+b+c)!}{a!b!c!}.$$

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41. (**An Unusual Inequality**) Let $x_0 = 0 < x_1 < x_2 < \cdots$. Show that

$$\sum_{n=1}^{\infty} \frac{x_n - x_{n-1}}{x_n^2 + 1} < \frac{\pi}{2}.$$

42. (**Hilbert's Inequality**) Let $(a_n)_1^\infty$ and $(b_n)_1^\infty$ be square-summable sequences of real numbers: $\sum_n a_n^2 < \infty$ and $\sum_n b_n^2 < \infty$. Show that

$$\sum_{m,n} \frac{a_m b_n}{m+n} < \pi \sqrt{\sum_m a_m^2} \sqrt{\sum_n b_n^2}.$$

43. (**The Size of the Central Binomial Coefficient**) Let $k \geq 1$ be an integer and $c, d > 0$ positive real numbers such that

$$\frac{c}{\sqrt{k-1/2}}\, 4^k \leq \binom{2k}{k} \leq \frac{d}{\sqrt{k+1/2}}\, 4^k.$$

Show that then the analogous inequalities hold for all $n \geq k$:

$$\frac{c}{\sqrt{n-1/2}}\, 4^n \leq \binom{2n}{n} \leq \frac{d}{\sqrt{n+1/2}}\, 4^n$$

whenever $n \geq k$. In particular,

$$\binom{2n}{n} < \begin{cases} 2^{2n-1} & \text{if } \ n \geq 2, \\ 2^{2n-2} & \text{if } \ n \geq 5, \end{cases}$$

and

$$\frac{0.5}{\sqrt{n-1/2}}\, 4^n \leq \binom{2n}{n} \leq \frac{0.6}{\sqrt{n+1/2}}\, 4^n$$

for $n \geq 4$.

44. (**Properties of the Central Binomial Coefficient**) Consider the prime factorization of the central binomial coefficient $\binom{2n}{n}$ for $n \geq 1$:

$$\binom{2n}{n} = \prod_{p<2n} p^{\alpha_p},$$

where p denotes a prime. Show the following assertions:

(i) $\alpha_p = 0$ or 1 if $\sqrt{2n} < p < 2n$;
(ii) $\alpha_p = 0$ if $2n/3 < p \leq n$;
(iii) $p^{\alpha_p} \leq 2n$ for every p.

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