"This is an excellent introduction to mathematical logic for upper-year undergraduate students with some background in theoretical mathematics. The many wonderful examples relate mathematical logic to other areas of mathematics. Beyond the usual setup and culmination with completeness and incompleteness, Mileti gives a strong introduction to set theory, model theory, and computability. Instructors have the flexibility to suit the examples and direction to their students' backgrounds, and curious students will have an excellent reference for topics not covered in their course. Mileti's gentle prose guides the reader from topic to topic, explaining what we are doing, and why."
Barbara F. Csima, University of Waterloo

"This book is a modern introduction to mathematical logic for a new generation of students. Filled to the brim with examples to motivate students and better explain the myriad topics covered, it is more detailed and complete than almost any text on the subject. Yet, amazingly, it is still quite concise and easy to read. Without a doubt, this is bound to be a new classic."
Damir Dzhafarov, University of Connecticut

"Building on its author's substantial experience teaching courses in the field, this book provides a thorough and wide-ranging introduction to mathematical logic. The careful choice of topics and high level of detail, particularly in early sections covering material that is often challenging even for mathematically well-prepared students, will make it suitable both for one-course introductions to the field and for course sequences aimed at preparing students for graduate-level courses in subfields such as set theory, model theory, computability theory, and proof theory."
Denis Hirschfeldt, University of Chicago

"Mileti's *Modern Mathematical Logic* is an absorbing, comprehensive, and well-organized textbook. It contains clear-cut and detailed descriptions of the basic ideas and mathematical arguments, a good number of helpful examples and stimulating exercises, and lucid explanations of the modern view of mathematical logic. It is highly recommended for beginners."
Makoto Kikuchi, Kobe University

"This is a great book. I've used drafts to teach mathematical logic many times in the past and students love it. They find that it is very complete, that it doesn't miss any detail, and that it is a great complement to my informal style of teaching."
Antonio Montalbán, University of California, Berkeley

"I enjoyed using early versions of this text. For mathematics students, it was much better than any of the standard texts. It begins with a general mathematical approach to induction, recursion, and generation to develop both syntax and semantics. It then offers a variety of tailored routes through wide swaths of logic using real mathematical examples. I highly recommend it."
Richard A. Shore, Cornell University

"This book gives a great introduction to mathematical logic for an advanced undergraduate or early graduate-level course. From propositional logic and completeness, through the basics of set theory and model theory, to the incompleteness theorem, the author skillfully presents notions and proofs without shying away from technical details. The exercises after each chapter are at the appropriate level, leading students into more in-depth explorations of various directions related to the topic at hand."
Mariya Soskova, University of Wisconsin–Madison

"This is the logic book I would have written if I could write as clearly as Mileti. At last, a complete introduction to the subject from a modern mathematical perspective that proves the incompleteness theorems while also providing a gentle introduction to model theory, set theory, and computability theory along the way."
Henry Towsner, University of Pennsylvania

"Mileti has written a careful, rigorous treatment of a broad range of topics in mathematical logic. This is a valuable addition to the literature."
Daniel J. Velleman, Amherst College

Modern Mathematical Logic

This textbook gives a complete and modern introduction to mathematical logic. The author uses contemporary notation, conventions, and perspectives throughout, and emphasizes interactions with the rest of mathematics. In addition to covering the basic concepts of mathematical logic and the fundamental material on completeness, compactness, and incompleteness, it devotes significant space to thorough introductions to the pillars of the modern subject: model theory, set theory, and computability theory.

Requiring only a modest background of undergraduate mathematics, the text can be readily adapted for a variety of one- or two-semester courses at the upper-undergraduate or beginning-graduate level. Numerous examples reinforce the key ideas and illustrate their applications, and a wealth of classroom-tested exercises serve to consolidate readers' understanding. Comprehensive and engaging, this book offers a fresh approach to this enduringly fascinating and important subject.

Joseph Mileti is Associate Professor in the Department of Mathematics and Statistics at Grinnell College, USA. He won the 2004 Sacks Prize from the Association of Symbolic Logic for the best doctoral dissertation in mathematical logic (worldwide) and leading experts have celebrated his original courses and materials taught at many universities.

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Contents

Preface

The field of mathematical logic is broad, and the subject plays an important role in mathematics, computer science, and philosophy. Although this book touches on each of these disciplines, it is designed for mathematics students at the advanced undergraduate or beginning graduate level. The book assumes familiarity with the material in a typical first course in Abstract Algebra, including comfort with reading and writing mathematical proofs. Additional mathematical background (particularly some combinatorics and analysis) is helpful but is not strictly necessary.

Approach of This Text

Mathematical logic has a bit of an image problem within mathematics. It is common for people to believe that the subject is about pathological objects, weirdness, and strange self-reference. Or that logicians primarily care about setting up a careful and rigorous foundation. However, like most mathematicians, mathematical logicians seek to find structure, beauty, and interactions with other areas of mathematics.

One of my main goals in writing this book has been to emphasize the mathematical side of mathematical logic. In addition to approaching the subject like any other area of mathematics, the book includes connections with random graphs, algebraically closed fields, and the structure of closed sets of reals. I have also chosen to include many algebraic examples to help illustrate the key ideas and theorems. While these choices impact the prerequisite assumptions of the reader, they do help make the subject feel significantly more alive and relevant.

Another major goal of the book is to treat the subject of mathematical logic as a unified and coherent whole. Instead of racing to the incompleteness theorems, the book develops the fundamentals of model theory, set theory, and computability theory along the way. While proof theory does not receive the same level of development, I have chosen to use a natural deduction proof system in both propositional and first-order logic. By taking the syntactic side seriously in the propositional logic case, the Completeness and Compactness Theorems in that context do prepare the reader for the more sophisticated first-order logic versions.

While I work out many technical (and at times a bit painful) constructions in detail, the book slowly relaxes the level of formalism over time. I try to avoid being bogged down in unnecessary details, but I do believe that it is important for newcomers to see how to reason about and manipulate the objects of study, especially because most readers likely have little experience working with syntactic objects. I struggled with these ideas when I was learning the subject, as many sources quickly pivoted to less careful definitions and arguments before I had mastered the toolkit to do the same. I hope that the book does develop the confidence necessary for readers to write their own arguments, but also helps them transition to more advanced work, where the formalism is often put into the background since it is (rightfully) viewed as routine and uninteresting.

One place where I do go into more detail than most sources is with respect to recursive definitions. I introduce a general framework for induction and recursion in Chapter 2 and then fit (essentially) all finitary recursive constructions into this framework. When I teach the course, I cover that chapter quickly, and leave most of the details for the curious and engaged reader. I would have greatly benefited from such a treatment as a student, but I understand that many others can find that level of detail and generality a bit dry and would rather move on to Chapter 3 as quickly as possible. Of course, there is no one correct way to proceed, but I have found that the ability to view recursion in this general way really helps the transition to transfinite recursion in Chapter 9 and also to a better understanding of primitive recursion in Chapter 11.

I have chosen to define the computable functions in two ways: one via (inductive) generation, and another using a machine model. By taking the time to understand and work with these two approaches and to prove their equivalence, the core intuition behind the Church–Turing thesis becomes apparent. Of course, the consequence of taking such an approach is that the section on computability theory is considerably longer than what one typically finds in an introductory logic book. However, after taking the time to explore computability carefully, the incompleteness and undecidability results become significantly more streamlined, transparent, and digestible. Moreover, by introducing various coding schemes in the context of computability theory, the use of Gödel numbers to code formulas becomes completely natural, as it is just one more example of a routine coding.

For set theory, I have found that most sources choose one of two approaches: either a naive treatment that does not assume any background in formal logic, or an advanced book that assumes complete comfort and races through the early material. I hope that Chapter 8 provides a helpful middle ground here. It takes the time to show how to encode mathematics within first-order set theory, while emphasizing the consequences of such an approach. In particular, it slowly introduces the reader to the shocking (to the uninitiated) fact that there are many models of the usual axioms of mathematics.

Flexible Organization and Coverage

The book includes more material than one finds in a typical one-semester logic course. I have successfully covered the vast majority of the book across two quarters at the University of Chicago, and most of the first 10 chapters in a one-semester course at Grinnell College. In the latter case, that means I chose to omit the incompleteness theorems. However, there are many other paths available. Chapter 3 through Chapter 6 is central to the book (with the possible exception of Section 6.6), but the following three blocks are then essentially independent of each other:

- Chapter 7 on model theory.
- Chapters 8 and 9 on set theory.
- Chapters 11 and 12 on computability and incompleteness.

Thus, after completing the core material through Section 6.5, one can proceed in any of the following ways:

1. A traditional and efficient course that wants to reach the most important incompleteness and undecidability results can immediately jump to Chapter 11 and then cover the first three sections of Chapter 12.
2. For a more enhanced version of the previous course that expands on the incompleteness and undecidability results, one can add the first two sections of Chapter 7 and then also include the last two sections of Chapter 12. While Section 7.2 is not a strict prerequisite for Section 12.4, I have found that some comfort and experience working with nonstandard models certainly helps one reason about representable functions and relations.
3. A course that is willing to trade away the incompleteness results for some set theory can jump to Chapter 8 and then continue through Chapter 9. From here, Chapter 10 is a nice capstone that applies the set-theoretic work back to logic, but some of the results there do rely on the work in Section 7.1. More model theory certainly helps to put the ideas in context but is not strictly necessary.

Exercises

Exercises appear at the end of each chapter but are grouped by (and labeled with) the corresponding section of the text. They vary in difficulty from routine verifications in order to build comfort with the definitions, to interesting applications of the core theorems, to more challenging problems (that either use more sophisticated mathematical ideas or extend the core ideas of the book). The more difficult problems and the ones that require additional mathematical knowledge are labeled with either (*) or (**).

Notation

I should also note that the book adopts a few pieces of nonstandard notation. In first-order logic, I keep a distinction between $\varphi(\mathtt{x}_1, \dots, \mathtt{x}_\mathtt{n})$, which is just used to introduce a formula with its free variables visible, and $\varphi[t_1, \dots, t_n]$, which means the result of substituting the corresponding terms for the variables. Since we can substitute the variables for themselves, I sometimes write $\varphi[\mathtt{x}_1, \dots, \mathtt{x}_\mathtt{n}]$ to emphasize the free variables when actually using the formula. I also keep semantic and syntactic notions completely separate between the sides of $\vDash$ when discussing semantic truth in a structure. For example, given an $\mathcal{L}$-structure $\mathcal{M}$ and $a_1, \dots, a_n \in M$, I write $(\mathcal{M}, a_1, \dots, a_n) \vDash \varphi$ instead of something like $\mathcal{M} \vDash \varphi[a_1, \dots, a_n]$.

Acknowledgments

This book began as a collection of course notes that I developed when I first started teaching mathematical logic in 2005. Over the last 15 years, I have worked on them across several courses and independent studies. Throughout all these stages, my students have helped me in so many ways. Their questions, feedback, and corrections over the years have significantly improved the structure and quality of the book. I heartily thank them all for their time and enthusiasm as well as for their patience with early drafts.

Damir Dzhafarov, Denis Hirschfeldt, Antonio Montalbán, Richard Shore, and Mariya Soskova used early drafts in their courses, and they each provided feedback and helpful encouragement along the way. I would also like to thank my thesis adviser, Carl Jockusch, for teaching me so many things about mathematical logic and for playing an important role in the early stages of my career.

Cambridge University Press supported me throughout the process and were clearly invested in my success. Several of the reviewers provided a number of thoughtful comments, ideas, and suggestions that I have incorporated.

Finally, I cannot imagine that this book would have been completed without the encouragement and support of my wife Jen. She gently pushed me to complete it, was a regular sounding board, and believed in me throughout. Many obstacles were thrown at us over the past couple of years (including actual trees), but she steadfastly supported me in this project.