Entropy and Diversity

The global biodiversity crisis is one of humanity's most urgent problems, but even quantifying biological diversity is a difficult mathematical and conceptual challenge. This book brings new mathematical rigour to the ongoing debate. It was born of research in category theory, is given strength by information theory, and is fed by the ancient field of functional equations. It applies the power of the axiomatic method to a biological problem of pressing concern, but it also presents new theorems that stand up as mathematics in their own right, independently of any application. The question 'what is diversity?' has surprising mathematical depth, and this book covers a wide breadth of mathematics, from functional equations to geometric measure theory, from probability theory to number theory. Despite this range, the mathematical prerequisites are few: the main narrative thread of this book requires no more than an undergraduate course in analysis.

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Entropy and Diversity

The Axiomatic Approach

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Much of this book was written in Catalonia during the years 2017 to 2019.

I dedicate it to all those who defend democracy.

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Some parts of Section 6.3 first appeared in Leinster and Meckes [221], and are reproduced with the second author's permission.

Note to the Reader

This book began life as a seminar course on functional equations at the University of Edinburgh in 2017, motivated by recent research on the quantification of biological diversity. The course attracted not only mathematicians in subjects from stochastic analysis to algebraic topology, but also participants from physics and biology. In response, I did everything I could to minimize the mathematical prerequisites.

I have tried here to retain the broad accessibility of the course. At the same time, I have not censored myself from including the many fruitful connections with more advanced parts of mathematics.

These two opposing forces have been reconciled by confining the more advanced material to separate chapters or sections that can easily be omitted. Chapter 9 requires some probability theory, Chapter 11 some abstract algebra, and Chapter 12 some category theory, while Sections 3.4, 6.4 and 6.5 also call on parts of geometry, analysis and statistics. However, the core narrative thread requires no more mathematics than a first course in rigorous (ε-δ) analysis. Readers with this background are promised that they are equipped to follow all the main ideas and results. The parts just listed, and any remarks that refer to more specialized knowledge, can safely be omitted.

Moreover, those who regard themselves as wholly 'pure' mathematicians will find no barriers here. Although much of this book is about the diversity of ecological systems, no knowledge of ecology is needed. Similarly, the information theory that we use is introduced from the ground up.

In the middle parts of the book, many conditions on means and diversity measures are defined: homogeneity, consistency, symmetry, etc. Appendix B contains a summary of this terminology for easy reference. There is also an index of notation.

Interdependence of Chapters

A dotted line indicates that one chapter is helpful, but not essential, for another.