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POLYNOMIAL METHODS AND INCIDENCE THEORY

The past decade has seen numerous major mathematical breakthroughs for topics such as the finite field Kakeya conjecture, the cap set conjecture, Erdös's distinct distances problem, the joints problem, as well as others, thanks to the introduction of new polynomial methods. There has also been significant progress on a variety of problems from additive combinatorics, discrete geometry, and more. This book gives a detailed yet accessible introduction to these new polynomial methods and their applications, with a focus on incidence theory.

Based on the author's own teaching experience, the text requires a minimal background, allowing graduate and advanced undergraduate students to get to grips with an active and exciting research front. The techniques are presented gradually and in detail, with many examples, warm-up proofs, and exercises included. An appendix provides a quick reminder of basic results and ideas.

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Polynomial Methods and Incidence Theory

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Introduction

Algebra is the offer made by the devil to the mathematician. The devil says: I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvellous machine.

Michael Atiyah (2005).

In his famous essay on how to write mathematics, Paul Halmos (1970) states, "Just as there are two ways for a sequence not to have a limit (no cluster points or too many), there are two ways for a piece of writing not to have a subject (no ideas or too many)." The book that you are now starting has two main subjects, which is hopefully a reasonable amount. These two subjects, *the polynomial method* and *incidence theory*, are tied together and difficult to separate.

Geometric incidences are a family of problems that have existed in discrete geometry for many decades. Starting around 2009, these problems have been experiencing a renaissance. New and interesting connections between incidences and other parts of mathematics are constantly being exposed. Incidences already have a variety of applications in harmonic analysis, theoretical computer science, model theory, number theory, and more. At the same time, significant progress is being made on long-standing open incidence problems. The study of geometric incidences is currently an active and exciting research field. One purpose of this book is to survey this field, the recent developments in it, and its connections to other fields.

What are incidences? Consider a set of points $\mathcal{P}$ and a set of lines $\mathcal{L}$ in the plane $\mathbb{R}^2$. An *incidence* is a pair $(p, \ell) \in \mathcal{P} \times \mathcal{L}$ such that the point p is on the line ℓ. For example, see Figure 1. One fundamental incidence result states that n points and n lines in $\mathbb{R}^2$ form at most $2.5n^{4/3}$ incidences. While the exponent $4/3$ cannot be improved, it is possible that the coefficient 2.5 could be replaced with a slightly smaller one.

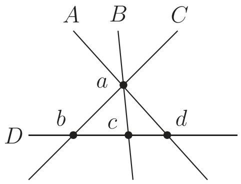

Figure 1 A configuration of four points, four lines, and nine incidences. For example, the point a forms an incidence with each of the lines A, B, and C.

In other incidence problems, we replace the lines with circles, parabolas, or other types of curves. Additional variants include incidences with higher-dimensional objects in $\mathbb{R}^d$, incidences with semi-algebraic sets, incidences with complex objects in $\mathbb{C}^d$, in spaces over finite fields, o-minimal structures, and more. In most of these cases, finding the maximum possible number of incidences remains an open problem.

An incidence result of a different flavor states that there exists a positive constant $c \in \mathbb{R}$ that satisfies the following. For every sufficiently large n, every set of n points in $\mathbb{R}^2$ satisfies at least one of the following statements:

- There exists a line that is incident to at least cn of the points.
- There exist at least cn^2 lines that are incident to at least two of the points.

Sylvester (1868) studied incidence problems back in the 1860s. The earliest incidence problem that we are aware of appears in a book of riddles (Jackson, 1821). This book contains 10 problems of the form that is presented in Figure 2. In modern English, the problem in Figure 2 asks for the following: Place points in the plane, such that the number of lines that contain exactly three points is at least the number of points.

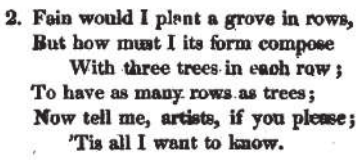
2. Fain would I plant a grove in rows,
But how must I its form compose
With three trees in each row;
To have as many rows as trees;
Now tell me, artists, if you please;
'Tis all I want to know.

Figure 2 A riddle from the 1821 book *Rational Amusement for Winter Evenings, Or, A Collection of Above 200 Curious and Interesting Puzzles and Paradoxes Relating to Arithmetic, Geometry, Geography.*

Most of the recent progress in incidence theory is due to new algebraic techniques. One may describe the philosophy behind these techniques as

Collections of objects that exhibit extremal behavior often have hidden algebraic structure. This algebraic structure can be exploited to gain a better understanding of the original problem.

For example, in a point-line configuration with many incidences, we might expect the points to form a lattice structure. Intuitively, we expose the algebraic structure by defining polynomials according to the problem, and then studying properties of these polynomials. In an incidence problem, we might study a polynomial that vanishes on all the points. This approach is called *the polynomial method*. In this book, we explore a wide variety of such polynomial proofs. We use these techniques to study incidence bounds, the finite field Kakeya problem, the cap set problem, distinct distances problems, the joints problem, and more.

Polynomial methods have existed for several decades. One well-known polynomial method is Alon's Combinatorial Nullstellensatz, as described in Alon (1999). As long ago as 1970, Rédei introduced an elegant polynomial proof. This book is focused on the new wave of polynomial methods that started to appear around 2009. These methods are quite different from the preceding ones.

This book aims to be an accessible introduction to the new polynomial methods and to incidence theory. For that reason, the book includes many examples, warm-up proofs, figures, and intuitive ways of thinking about tricky ideas. Many techniques are presented gradually and in detail. Readers who wish to dig deeper into a particular topic can find references in the relevant chapter.

Incidence theory and the polynomial methods are still developing. There are many interesting open problems, and, in some sense, the foundations are not completely established yet. For that reason, most of the chapters of this book end with an open problems section. These sections focus mostly on long-standing difficult problems. Their goal is to illustrate the current research fronts and the main difficulties that researchers are currently facing.

Several sections are defined as *optional*. Some sections, such as Section 7.3, are optional because they consist of standard technical proofs that may not provide any new insights. Other sections require familiarity with a topic that is orthogonal to the topics of this book. For example, the optional Section 9.3 requires basic familiarity with differential topology, which does not appear anywhere else in the book.

Two other good sources for polynomial methods in discrete geometry are the book *Polynomial Methods in Combinatorics* (Guth, 2016) and the survey

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"Incidence theorems and their applications" (Dvir, 2012). While these sources and the current book study similar topics, the overlap between them is smaller than one might expect.

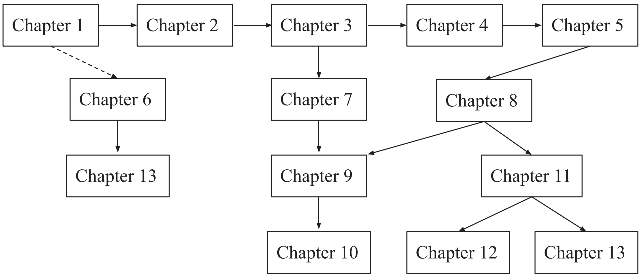

Figure 3 Chapter dependencies. The dashed edge marks a dependency that is recommended but not necessary.

How to Read This Book

Throughout this book, we rely heavily on asymptotic notation such as $x = O(y)$. The appendix contains an introduction to asymptotic notation, together with exercises. This appendix also briefly surveys basic graph theory notation and the Cauchy–Schwarz inequality.

There are many ways to read this book, depending on the goal of the reader. One way is to start from the beginning and read the chapters consecutively. The beginning of the book contains more introductory material. The end of the book contains mostly optional advanced topics. Figure 3 illustrates the chapter dependencies. Some reading options are:

- **A brief introduction to discrete geometry.** For an introduction to problems and techniques from classical discrete geometry, read Chapter 1. This chapter does not involve polynomial methods.
- **An introduction to polynomial partitioning.** To learn how to prove incidence results by using polynomial methods, read Chapters 1–3. Chapter 2 is a minimal introduction to algebraic curves in the real plane. Chapter 3 consists of the basics of the *polynomial partitioning* technique, and how to use this technique to prove incidence bounds.
- **A variety of polynomial methods in combinatorics.** To see a variety of polynomial methods in combinatorics, read Chapters 1–6. In addition to the polynomial partitioning technique, Chapters 5 and 6 contain several other polynomial breakthroughs. Chapter 4 introduces basic concepts from

real algebraic geometry, and can be quickly skimmed by a reader who does not intend to read beyond Chapter 6. Chapter 5 contains the polynomial proof of the joints theorem. Chapter 6 contains polynomial proofs for problems in finite fields, such as the finite field Kakeya problem and the cap set problem.

- **The distinct distances theorem.** To understand the distinct distances theorem of Guth and Katz, read Chapters 1–5 and 7–10. Chapter 7 reduces the distinct distances problem to an incidence problem in $\mathbb{R}^3$. Chapter 8 introduces the *constant-degree polynomial-partitioning* technique and uses it to prove incidence bounds in the complex plane. Chapter 9 extends this technique and uses it to prove the distinct distances theorem. Chapter 10 studies a few variants of the distinct distances problem.
- **Incidences and polynomial methods over finite fields.** To study incidences and polynomial methods over finite fields, read Chapters 6 and 13. You might wish to first read Chapter 1, but this is not necessary. Chapter 13 studies point-line incidences over finite fields.
- **Incidences in $\mathbb{R}^d$.** To understand advanced incidence techniques in $\mathbb{R}^d$, read Chapters 1–5, 8, 11, 12, and 14. Chapter 11 studies more advanced techniques for deriving incidence bounds in $\mathbb{R}^d$. Chapter 12 consists of applications for such incidence bounds. Chapter 14 introduces more advanced tools for studying incidences and related problems. In particular, this final chapter studies properties of ruled surfaces.

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