Reduction Theory and Arithmetic Groups

Arithmetic groups are generalisations, to the setting of algebraic groups over a global field, of the subgroups of finite index in the general linear group with entries in the ring of integers of an algebraic number field. They are rich, diverse structures, and they arise in many areas of study.

This text enables you to build a solid, rigorous foundation in the subject. It first develops essential geometric and number of theoretical components to the investigations of arithmetic groups and then examines a number of different themes, including reduction theory, (semi)-stable lattices, arithmetic groups in forms of the special linear group, unipotent groups and tori, and reduction theory for adelic coset spaces. Also included is a thorough treatment of the construction of geometric cycles in arithmetically defined locally symmetric spaces and some associated cohomological questions. Written by a renowned expert, this book will be a valuable reference for researchers and graduate students alike.

Joachim Schwermer is Emeritus Professor of Mathematics at the University of Vienna, and recently Guest Researcher at the Max-Planck-Institute for Mathematics, Bonn. He was Director of the Erwin-Schrödinger-Institute for Mathematics and Physics, Vienna, from 2011 to 2016. His research focuses on questions arising in the arithmetic of algebraic groups and the theory of automorphic forms.

NEW MATHEMATICAL MONOGRAPHS

All the titles listed below can be obtained from good booksellers or from Cambridge University Press.
For a complete series listing, visit www.cambridge.org/mathematics.

5. Y. J. Ionin and M. S. Shrikhande *Combinatorics of Symmetric Designs*
6. S. Berhanu, P. D. Cordaro and J. Hounie *An Introduction to Involutive Structures*
7. A. Shlapentokh *Hilbert's Tenth Problem*
8. G. Michler *Theory of Finite Simple Groups I*
9. A. Baker and G. Wüstholz *Logarithmic Forms and Diophantine Geometry*
10. P. Kronheimer and T. Mrowka *Monopoles and Three-Manifolds*
11. B. Bekka, P. de la Harpe and A. Valette *Kazhdan's Property (T)*
12. J. Neisendorfer *Algebraic Methods in Unstable Homotopy Theory*
13. M. Grandis *Directed Algebraic Topology*
14. G. Michler *Theory of Finite Simple Groups II*
15. R. Schertz *Complex Multiplication*
16. S. Bloch *Lectures on Algebraic Cycles (2nd Edition)*
17. B. Conrad, O. Gabber and G. Prasad *Pseudo-reductive Groups*
18. T. Downarowicz *Entropy in Dynamical Systems*
19. C. Simpson *Homotopy Theory of Higher Categories*
20. E. Fricain and J. Mashreghi *The Theory of H(b) Spaces I*
21. E. Fricain and J. Mashreghi *The Theory of H(b) Spaces II*
22. J. Goubault-Larrecq *Non-Hausdorff Topology and Domain Theory*
23. J. Śniatycki *Differential Geometry of Singular Spaces and Reduction of Symmetry*
24. E. Riehl *Categorical Homotopy Theory*
25. B. A. Munson and I. Volić *Cubical Homotopy Theory*
26. B. Conrad, O. Gabber and G. Prasad *Pseudo-reductive Groups (2nd Edition)*
27. J. Heinonen, P. Koskela, N. Shanmugalingam and J. T. Tyson *Sobolev Spaces on Metric Measure Spaces*
28. Y.-G. Oh *Symplectic Topology and Floer Homology I*
29. Y.-G. Oh *Symplectic Topology and Floer Homology II*
30. A. Bobrowski *Convergence of One-Parameter Operator Semigroups*
31. K. Costello and O. Gwilliam *Factorization Algebras in Quantum Field Theory I*
32. J.-H. Evertse and K. Győry *Discriminant Equations in Diophantine Number Theory*
33. G. Friedman *Singular Intersection Homology*
34. S. Schwede *Global Homotopy Theory*
35. M. Dickmann, N. Schwartz and M. Tressl *Spectral Spaces*
36. A. Baernstein II *Symmetrization in Analysis*
37. A. Defant, D. García, M. Maestre and P. Sevilla-Peris *Dirichlet Series and Holomorphic Functions in High Dimensions*
38. N. Th. Varopoulos *Potential Theory and Geometry on Lie Groups*
39. D. Arnal and B. Currey *Representations of Solvable Lie Groups*
40. M. A. Hill, M. J. Hopkins and D. C. Ravenel *Equivariant Stable Homotopy Theory and the Kervaire Invariant Problem*
41. K. Costello and O. Gwilliam *Factorization Algebras in Quantum Field Theory II*
42. S. Kumar *Conformal Blocks, Generalized Theta Functions and the Verlinde Formula*
43. P. F. X. Müller *Hardy Martingales*
44. T. Kaletha and G. Prasad *Bruhat–Tits Theory*

Reduction Theory and Arithmetic Groups

JOACHIM SCHWERMER
University of Vienna

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre,
New Delhi – 110025, India

103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781108832038
DOI: 10.1017/9781108937610

First published 2023

A catalogue record for this publication is available from the British Library.

Library of Congress Cataloging-in-Publication Data
Names: Schwermer, Joachim, author.
Title: Reduction theory and arithmetic groups / Joachim Schwermer.
Description: Cambridge ; New York, NY : Cambridge University Press, 2023. |
Series: New mathematical monographs | Includes bibliographical references and index.
Identifiers: LCCN 2022022858 | ISBN 9781108832038 (hardback) |
ISBN 9781108937610 (ebook)
Subjects: LCSH: Arithmetic groups. | Number theory. |
BISAC: MATHEMATICS / Number Theory
Classification: LCC QA174.2 .S349 2023 | DDC 512.7/4–dc23/eng20220902
LC record available at https://lccn.loc.gov/2022022858

ISBN 978-1-108-83203-8 Hardback

Cambridge University Press & Assessment
978-1-108-83203-8 — Reduction Theory and Arithmetic Groups
Joachim Schwermer
Frontmatter
More Information

Contents

Preface

Arithmetic groups are generalisations, to the setting of algebraic groups defined over a global field, of the subgroups of finite index in the general linear group $\mathrm{GL}_n(\Lambda)$ with entries in the ring of integers Λ of an algebraic number field k or, more generally, an order Λ in a finite-dimensional division algebra over k. Historically such groups arose naturally in the study of arithmetic properties of quadratic forms. The study of reduction of such forms, as developed by Gauss, Hermite, and Minkowski, among others, gave a powerful way to select, from the infinitely many forms, that are integrally equivalent to a given form, one that is intrinsically characterised by suitable conditions on its entries. Minkowski, following a suggestion made by Gauss in 1831, created a new version of reduction theory by working with lattices as geometric objects. His works, especially his geometric point of view, served as substantial stimuli for Siegel's studies of quadratic, symplectic, or Hermitian forms and their associated discontinuous groups. Since the development of the general theory of linear algebraic groups over fields, it has been natural to view arithmetic groups as a rich integral extension of that algebro-geometric theory. Thus, thanks to the work of Chevalley, Borel, Serre, Harder, and Raghunathan, the study of arithmetic groups today can start from the theory of algebraic groups defined over a field k, which is either an algebraic number field or a finite separable extension of $F_q(t)$, where F_q is a finite field and t is transcendental over F_q, i.e. k is a global field.

An arithmetic group Γ acts on a homogeneous space which is defined by the ambient algebraic group. This action and the study of the orbit space are of interest, both intrinsically and for the insight into the structure of Γ. Thus, there is an essential geometric component to the investigations of these groups. Further, arithmetic groups arise in a wide variety of mathematical contexts, ranging from differential geometry, in particular, the theory of locally symmetric spaces, topology, geometric group theory, to number theory and arithmetic algebraic geometry, the theory of automorphic forms over global fields, and even lately quantum computing.

Therefore, on the one hand, ‘Arithmetic Groups’ do not present themselves as a coherent limited theory, so that a book dealing with this area faces a challenge. On the other hand, there are some overarching results and constructions, in particular, regarding reduction theory, which are fundamental to many of the areas discussed above and which are likely to be used in different contexts. Thus, as a resolution, inspired by the approach in Carl E. Schorske’s *Fin-de-Siècle Vienna*, each chapter of this monograph is ‘issued from a separate foray into the terrain, varying in scale and focus according to the nature of the problem’.

It is intended to lay rigorous and solid groundwork in dealing with $\mathcal{S}$-arithmetic groups in algebraic groups defined over a global field k. At the outset, we examine the fundamental case of the general linear group. Additionally, we survey some of the historical sources, which might be helpful for understanding the genesis of this mathematical area.

We begin with an analysis of the normal subgroup structure of the general linear group GL_n over a (non-commutative) ring with identity and an introduction of the basic concepts regarding $\mathcal{S}$-arithmetic groups in GL_n. By definition, the ring $O_{k,\mathcal{S}}$ of $\mathcal{S}$-integers in a global field k, associated with a finite set $\mathcal{S}$ of places of k which includes the archimedean ones in the case of an algebraic number field, is the ring of elements of k integral at each place outside of $\mathcal{S}$. If k is an algebraic number field and $\mathcal{S}$ consists only of the archimedean places of k, the ring of $\mathcal{S}$-integers in k is the usual ring of integers O_k in k. An $\mathcal{S}$-arithmetic subgroup of $GL_n(k)$ is defined to be a subgroup which is commensurable with $GL_n(O_{k,\mathcal{S}})$. Any ideal $\mathfrak{q}$ in $O_{k,\mathcal{S}}$ gives rise to the principal $\mathcal{S}$-congruence subgroup $\mathrm{GL}_n(O_{k,\mathcal{S}}, \mathfrak{q})$ of level $\mathfrak{q}$. It is defined as the kernel of the group homomorphism $\mathrm{GL}_n(O_{k,\mathcal{S}}) \longrightarrow \mathrm{GL}_n(O_{k,\mathcal{S}}/\mathfrak{q})$, thus a normal subgroup of finite index in the $\mathcal{S}$-arithmetic group $\mathrm{GL}_n(O_{k,\mathcal{S}})$. Any $\mathcal{S}$-arithmetic subgroup that contains a principal $\mathcal{S}$-congruence subgroup for some ideal $\mathfrak{q}$ is called a congruence subgroup. We indicate how $\mathcal{S}$-arithmetic groups, depending on the form of $\mathcal{S}$, can be naturally viewed as discrete subgroups in a reductive Lie group, real, p-adic or product of such groups, to be denoted $\prod_{v\in\mathcal{S}} G_v =: G_{\mathcal{S}}$. For each place $v \in \mathcal{S}$, there is a corresponding homogeneous space X_v, and an $\mathcal{S}$-arithmetic subgroup, viewed as a discrete subgroup of $G_{\mathcal{S}}$, naturally acts on the product $X_{\mathcal{S}} := \prod_{v\in\mathcal{S}} X_v$. The resulting orbit spaces are the objects that concern us. Having these essentials in place, we follow different thematic branches. As a conclusion to the first part of the book, pointing towards the geometric perspective, we discuss reduction theory in the case of arithmetically defined subgroups of the group of orientation preserving isometries of hyperbolic 3-space and study the orbit spaces.

The second part begins with the uniform construction of Siegel sets in the case GL_n over the basic cases of a global field, namely, the field $\mathbb{Q}$ of rational numbers or the field $F_q(t)$ of rational functions in the variable t and having coefficients in the finite field F_q. This chapter develops the theory of successive minima in each case. In the following chapter, we look at reduction theory for GL_n in a different manner. We define the notion of (semi)-stability for Euclidean $\mathbb{Z}$-lattices, or, more generally, for arithmetic O_k-lattices, and construct their immanent canonical filtration.

Then, in Chapter 8, we take up the general case of an $\mathcal{S}$-arithmetic subgroup of an algebraic group over a global field k. To illustrate the various notions, we deal with important families of examples: we carry out Chevalley's construction of group schemes over $\mathbb{Z}$, which give rise to $\mathbb{Q}$-split reductive algebraic $\mathbb{Q}$-groups, endowed with a natural integral structure. Besides the general making of these $\mathbb{Z}$-group schemes, the cases that originate from root systems (and related weight lattices) of type A_{n-1} and C_n are dealt with in detail. Next, in the case of an algebraic number field, we associate with a given maximal O_k-order M in a central simple k-algebra A an affine smooth O_k-group scheme SL_M of finite type. This construction provides integral structures on the algebraic k-group SL_A, an inner form of the special linear group. This is supplemented by a discussion of outer forms, namely, special unitary groups attached to division algebras, equipped with an involution of the second kind, and suitable arithmetic subgroups. We finally exhibit various applications of the local–global principle encoded in the Brauer–Hasse–Noether theorem and the corresponding exact sequence of Brauer groups attached to k. This concerns the construction of central division algebras with specific local behaviour. These results allow us to exhibit different k-forms of the special linear group.

As indicated in the case of the general linear group, $\mathcal{S}$-arithmetic subgroups Γ of algebraic k-groups give rise to geometric objects, e.g. symmetric spaces and Bruhat–Tits buildings alike, and products of such, on which $\mathcal{S}$-arithmetic groups act. The corresponding orbit spaces form critical components of the study of the $\mathcal{S}$-arithmetic groups. In Chapter 9, we introduce these spaces and exhibit some of their properties. Then, with a focus on the case of an algebraic number field, and $\mathcal{S}$ is equal to the set of archimedean places of k, we study arithmetic groups in unipotent groups and algebraic k-tori. Next we treat Godement's compactness criterion for the orbit space $\Gamma\backslash X$ attached to an arithmetic group Γ in an algebraic k-group G and its action on $X := X_{\mathcal{S}}$. Consequences here include the fact that the image of an arithmetic group under a surjective morphism of algebraic k-groups is an arithmetic group. As an application of the criterion, we are in the position to construct in detail various families of examples, both compact and non-compact, of orbit spaces.

In Chapters 10 and 11, given an arithmetic subgroup of a connected reductive k-group, we go more deeply into the geometry of the orbit space $\Gamma\backslash X$, endowed with a Riemannian structure. We focus on constructing totally geodesic cycles in $\Gamma\backslash X$

which originate from reductive subgroups H of G. More precisely, in an obvious notation, a reductive k-subgroup H of G gives rise to a natural map

$$j_{H|\Gamma}\colon (\Gamma \cap H(k))\backslash X_H \longrightarrow \Gamma\backslash X.$$

The basic results, which are proved in Chapter 10, guarantee that this map (by passing to a finite covering if necessary) is a proper, injective, closed embedding, and so that each connected component of the image is an orientable, totally geodesic submanifold of $\Gamma\backslash X$. This requires us to treat critical questions regarding orientability. In many cases, it will be shown that these cycles, to be called *geometric cycles*, yield non-vanishing (co)homology classes for the underlying orbit space $\Gamma\backslash X$. This result is based on an analysis of the intersection number of a given geometric cycle with a suitably chosen geometric cycle of complementary dimension. Unfortunately, geometric cycles of complementary dimension usually intersect in a complicated set, possibly of dimension greater than zero. To handle this situation, the theory of 'excess intersections' has to be introduced. Under suitable conditions, the intersection number of two such cycles can be expressed as the sum of the Euler numbers of the excess bundles corresponding to the connected components of the intersection.

In Chapter 12, we state the core results in reduction theory for the adelic points of a connected reductive algebraic group defined over a global field k. The standard methods of proof (for which we refer to the literature) depend on whether k is an algebraic number field or a function field. We sketch a uniform approach in the case of split k-groups, which rests on the concept of adelic heights.

We conclude this preface with a personal note: against all expectations, there are only two passages in the book where we talk about cohomology of arithmetic groups. First, in the case of arithmetic groups Γ in unipotent $\mathbb{Q}$-groups U, one finds in Section 9.3 an algebraic proof of the result that the cohomology groups $H^*(\Gamma, M)$ and $H^*(U, M)$ are isomorphic for any finite-dimensional U-module M. This is a rational version of van Est's theorem. Second, Section 11.6 gives a short outlook on how the investigations of geometric cycles and the analysis of suitable intersection numbers can be used to obtain non-vanishing results for the cohomology of arithmetic groups.

As one can see from the table of contents, this book comprises two parts, distinguishable from one another by their level of sophistication, technical understanding required, and essential prerequisites that are assumed from the reader. In particular, Part II is based on a solid familiarity with the theory of algebraic k-groups. As an additional aid for readers, we include three Appendices, filling in some background material. Appendix A reviews main notions and results from the theory of affine group schemes and algebraic k-groups. Appendix B is a brief account of the definitions and key results regarding global fields; it also serves to fix notations for the text. Finally, due to a lack of a suitable reference, Appendix C compiles (with complete proofs) basic facts concerning proper actions of topological groups on topological spaces, with special emphasis on discrete groups.

In view of the richness of the mathematical terrain of arithmetic groups, the account we give can only be very selective in its choices and can only touch upon its most salient characteristics. Nevertheless, we hope that the reader will be able to obtain a reasonably detailed understanding of this area of research and to experience its beauty.

We regret that we had to exclude various, even basic, results as beyond the scope of this manuscript. However, the interested reader will find open questions and directions for further research or study scattered in the text.

Acknowledgements I am grateful to all my students and graduate students over the years in Vienna whose valuable questions and insights in our seminars helped to convince me of the value of writing this book. In particular, I owe a great debt to Gerald Gotsbacher, Ognjen Vukadin, Sascha Biberhofer, Susanne Schimpf, Steffen Kionke, Sophie Koch, and Johannes Girsch.

First and foremost, Stefan Kühnlein and Joachim Mahnkopf devoted considerable time and effort in reading the manuscript. I gratefully acknowledge their valuable help and advice both in detecting errors or inaccuracies and suggesting improvements.

Furthermore, I am grateful for encouragement and comments from many people who read and commented on earlier versions of individual chapters of this work. For special mention, I must single out Jens C. Jantzen, Ulrich Stuhler, James Cogdell, Don Blasius, Hayuk Sengun, and Tony Scholl.

I gratefully acknowledge the support of the Max Planck Institute for Mathematics, Bonn, in the academic years 2019–20 and 2020–21 where much of the work on this book was carried out.