The Discrete Mathematical Charms of Paul Erdős

Paul Erdős published more papers during his lifetime than any other mathematician, especially in discrete mathematics. He had a nose for beautiful, simply stated problems with solutions that have far-reaching consequences across mathematics. This captivating book, written for students, provides an easy-to-understand introduction to discrete mathematics by presenting questions that intrigued Erdős, along with his brilliant ways of working toward their answers. It includes young Erdős's proof of Bertrand's postulate, the Erdős–Szekeres Happy End Theorem, De Bruijn–Erdős theorem, Erdős–Rado delta-systems, Erdős–Ko–Rado theorem, Erdős–Stone theorem, the Erdős–Rényi-Sós Friendship Theorem, Erdős–Rényi random graphs, the Chvátal–Erdős theorem on Hamilton cycles, and other results of Erdős, as well as results related to his work, such as Ramsey's theorem or Deza's theorem on weak delta-systems. Its appendix covers topics normally missing from introductory courses. Filled with personal anecdotes about Erdős, this book offers a behind-the-scenes look at interactions with the legendary collaborator.

VAŠEK CHVÁTAL is Professor Emeritus of Concordia University, where he served as Canada Research Chair in Combinatorial Optimization (2004–11) and Canada Research Chair in Discrete Mathematics from 2011 until his retirement in 2014. He is the author of *Linear Programming* (1983) and co-author of *The Traveling Salesman Problem: A Computational Study* (2007). In the 1970s, he wrote three joint papers with Paul Erdős. He is a recipient of the CSGSS Award for Excellence in Teaching, Rutgers University (1992, 1993, 2001) and co-recipient of the Beale–Orchard–Hays Prize (2000), Frederick W. Lanchester Prize (2007), and John von Neumann Theory Prize (2015).

"Vašek Chvátal was born to write this one-of-a-kind book. Readers cannot help but be captivated by the evident love with which every page has been written. The human side of mathematics is intertwined beautifully with first-rate exposition of first-rate results."

– Donald Knuth, Stanford University

"This book is a treasure trove from so many viewpoints. It is a wonderful introduction and an alluring invitation to discrete mathematics – now a central field of mathematics identified mostly with the hero of this book. With lucid, carefully planned chapters on different topics, it demonstrates the unique way in which Paul Erdős, one of the most prolific and influential mathematicians of the 20th century, invented and approached problems. Sprinkled with historical and personal anecdotes and pictures, it opens a window to the unique personality of "Uncle Paul". And implicitly, it reveals the charming and candid way in which Vašek Chvátal, an authority in the field and a lifelong friend and collaborator of Erdős, likes to combine teaching and story-telling."

– Avi Wigderson, IAS, Princeton

"Paul Erdős is one of the founding fathers of modern combinatorics, whose ability to pose beautiful problems greatly determined the development of this field and influenced many other areas of mathematics. This book uses some basic questions, which intrigued Paul Erdős, to give a nice introduction to many topics in discrete mathematics. It contains a collection of beautiful results, covering such diverse subjects as discrete geometry, Ramsey theory, graph colorings, extremal problems for graphs and set systems and some others. It presents many elegant proofs and exposes the reader to various powerful combinatorial techniques."

– Benjamin Sudakov, ETH Zurich

"This is a brilliant book. It manages in one fell swoop to survey and develop a large part of combinatorial mathematics while at the same time chronicling the work of Paul Erdős. His contributions to different areas of mathematics are seen here to be part of a coherent whole. Chvátal's presentation is particularly appealing and accessible. The wonderful personal recollections add to the mathematical content to provide a portrait of Erdős' mind recognizable to those who knew him."

– Bruce Rothschild, University of California, Los Angeles

"Vašek Chvátal's book is a gem. Paul Erdős' favorite problems and best work are beautifully laid out. Readers unfamiliar with Erdős' work cannot fail to appreciate its power and elegance, and those who have seen bits and pieces will have the pleasure of seeing it thoughtfully and lovingly presented by a master. It's hard to imagine now, but there was a time when combinatorics was thought to be a jumble of results without depth or coherence. "Uncle Paul" understood its heart and soul, and nowhere is this more evident than in Chvátal's wonderful compendium. This volume belongs on every math-lover's night-table!"

– Peter Winkler, Dartmouth College

"Beautiful mathematics is presented with great care and clarity in Vašek Chvátal's book, complemented with well-written anecdotes and personal reminiscences about Paul Erdős. This combination makes the book a very enjoyable reading and a lively tribute to the memory of one of the most prolific mathematicians of all time. Studying discrete mathematics from this book is likely to give a great experience to students and established researchers alike."

– Gábor Simonyi, Hungarian Academy of Sciences

"A fantastic blend of math, history and personal anecdotes; a true mathematician's perspective on the legacy of a legend."

– Maria Chudnovsky, Princeton University

Vašek Chvátal, 1972. Photo©Adrian Bondy

The Discrete Mathematical Charms of Paul Erdős

A Simple Introduction

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To Markéta

Another roof, another proof.

Paul Erdős

There will be plenty of time to rest in the grave.

Paul Erdős

Nostalgia isn't what it used to be.

Simone Signoret

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Foreword

There will be an answer, let it be...

The Beatles

It is every scientist's dream to publish a work which is complete and perfect, which presents the best solution to a major problem or the ultimate survey of an important field of research. A work which will require no updates for a very long time. Unfortunately, this dream is almost never realized, for it would contradict the laws of the evolution of science. Sometimes the failure is dramatic. After completing his magnum opus, the *Grundgesetze,* in which he proposed a solid foundation for arithmetic based on mathematical logic, Gottlob Frege received a letter from Bertrand Russell, describing his famous paradox. Frege had to append an epilogue to his book, in which he wrote: "A scientific writer can hardly encounter anything more undesirable than that after completing his work, its foundations are shattered. I was put into this position by a letter from Mr. Bertrand Russell."[a]

Most cases, however, are more mundane. There are almost two hundred thousand mathematical papers published every year. Even if none of them shakes the foundation of a particular monograph in preparation, many may be relevant to the subject. A perfectionist author might want to reference at least the most significant ones. This is a never-ending struggle, and prolonging the process may actually harm the project. I have seen many beautiful mathematical manuscripts which, by the time they were published, doubled in size and became much less appealing. When Vašek sent me the first version of his book *The Discrete Mathematical Charms of Paul Erdős,* it struck me that this was not a draft. It was a piece of art, essentially complete and finished. I replied to him right away: "By adding more references, details, pointers, and clarifying paragraphs here and there, you would only blur this miraculous gem. Just correct the misprints and obvious errors and sit back! *Let it be!*" Vašek wrote back immediately. He was puzzled: How could I possibly have learned about the cover he proposed to his editor? I had no idea what he was talking about. Then I opened the attachment to his email, and it was my turn to be baffled: I saw a replica of the cover of the famous Beatles album, *Let It Be*, with the portraits of the four band members

[a] Einem wissenschaftlichen Schriftsteller kann kaum etwas Unerwünschteres begegnen, als dass ihm nach Vollendung einer Arbeit eine der Grundlagen seines Baues erschüttert wird. In diese Lage wurde ich durch einen Brief des Herrn Bertrand Russell versetzt, als der Druck dieses Bandes sich seinem Ende näherte [6].

replaced by four photos of Paul Erdős! We concluded that this must have been just another manifestation of Jung's synchronicity.

Most of the songs of *Let It Be* were conceived during the tumultuous year of 1968, which brought huge anti-war and civil rights protests in the United States, general strikes and student riots in France, as well as the Prague Spring, a major attempt to liberalize Soviet-type communism. Suddenly everything seemed possible: Yale College started admitting female students, the first manned spacecraft entered orbit around the moon, Pierre Trudeau became prime minister of Canada . . . and the 22-year-old Vašek Chvátal left Czechoslovakia to discover and conquer the world. He was about the same age as Paul Erdős at the time Erdős left Hungary for England to escape growing anti-Semitism and chauvinism in his country. In 1934, the political atmosphere appeared much grimmer than in 1968. Nevertheless, it is hard not to notice that the quest for personal freedom played a similar role in the lives of Erdős and Chvátal.

"Von Haus aus,"[b] Erdős spoke very good German and English, albeit with a heavy Hungarian accent. When George Csicsery made a documentary [5] about him, Erdős's English words were consistently subtitled in English. He did not pay much attention to polishing his pronunciation or style, but his systematic use of a special, Erdős-esque vocabulary added a unique and humorous flavor to his speech.[c] Vašek learned English and French in record time. Four years after "making landfall" in Canada,[d] his short story [3] was included in Martha Foley's list of Best American Short Stories of the year.

By offering a new "unorthodox" combinatorics course at Concordia University, Chvátal set out on a very ambitious project: to give a concise introduction to some of the basic concepts and results of discrete mathematics through the work of Paul Erdős, one of the founding fathers of the subject. The present book grew out of this project. What is the best way to present this material? Vašek is a master expositor. At the beginning of his classic monograph [4], he illustrated his approach by quoting Ralph P. Boas [2]: "Suppose you want to teach the 'cat' concept to a very young child. Do you explain that a cat is a relatively small, primarily carnivorous mammal with retractile claws, a distinct sonic output, etc.? I'll bet not. You probably show the kid a lot of different cats saying 'kitty' each time until it gets the idea." This strategy is in perfect agreement with the spirit of Erdős's mathematics. He did not like to formulate big theories and metatheorems. He had a nose for seductively beautiful, simply stated mathematical problems with solutions that have far-reaching consequences in several areas across mathematics. Each section of this book presents one or more such questions, together with Erdős's brilliant attempts to answer them. Each answer, each theorem, leads to a multitude of further exciting problems.

[b] Straight from home.
[c] "Slave" for husband, "boss" for wife, "epsilons" for children, "Sam" for the United States, "Joe" for the Soviet Union, "poison" for alcohol, etc.
[d] Adrian Bondy's expression from Section 1 of [1].

I must warn you: If you are attracted to combinatorics and the chemistry is right, you are likely to fall in love with this book. In any case, quoting from [3], "either you get the telepathic shock, the unmistakable click, and then you just KNOW it – the only thing you can be absolutely sure about, outside the cogito-ergo-sum syllogisms world. Or you don't and then there is no way to put it into words for you."

János Pach
Rényi Institute (Budapest) and EPFL (Lausanne)

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Preface

Paul Erdős (26 March 1913 – 20 September 1996) was an outstanding, prolific, influential, legendary mathematician.

Three times between January 2007 and December 2009, I taught at Concordia a one-term course of my own design entitled *Discrete Mathematics of Paul Erdős*. This was a graduate course, but it was open to undergraduates as well. Colleagues from other Montreal universities frequently sat in the audience, too, and this delighted me very much. The lecture rooms assigned to us were not always adequate. The sight of people huddled in the space overheated by their bodies and straining their ears in the doorway made me feel like a participant in clandestine gatherings of the early Christians. The present book is based on my lecture notes for that course. From time to time, I strayed away from the syllabus and talked about my own interactions with Erdős. A few of these recollections are recorded here, too.

Others were much closer to Erdős than I was. Others are better qualified to provide a tribute to him than I am (and some did). Still, large mosaics are made out of small fragments such as the reminiscences of mine that are sprinkled throughout the following text. I have been one of the blind men holding onto an elephant, and these vignettes form my report on what I felt.

The objective of my course was to survey results of Erdős and others that laid the foundations of discrete mathematics before it matured into the rich and vibrant discipline of today. Revisiting them after several decades brought back memories of pulling down heavy volumes from library shelves, leafing through their yellowing pages, and wondering at the treasures that they revealed. Memories like faded sepia photographs. Memories of a bygone era.

Paul Erdős had done a lot for me and enriched my life very much. Teaching that course and writing this book brought me to his presence again and sharpened my focus on him. They entailed my repeated silent thanks to him.

Acknowledgments

In 2015, Gösta Grahne told me about the website containing a declassified FBI file on Paul Erdős, and a few years later, Jermey Matthews told me about the book containing a part of this file. (Section C.5 of the Appendix refers to both.)

When I began turning my lecture notes into this book, I benefited from consultations with and help from Geoffrey Exoo, Nikolai Ivanov, Brendan McKay, Sergei Malyugin, and Staszek Radziszowski. A year later, Yogen Chaubey, Bipin Desai, and Tariq Srivastava joined me in tackling the mystery of the Majindar/Majumdar duality noted in Chapter 2.

Markéta Vyskočilová read early versions of the vignettes set here in sans serif, and her comments helped me very much in improving them. I am blessed to be married to such a thoughtful, discerning, brilliant editor.

Peter Renz guided me through the realm of publishing, and his advice has been invaluable to me.

Noga Alon, Laci Babai, Thomas Bloom, Xiaomin Chen, Javier De Loera, Mark Goldsmith, Peter Grogono, Steve Hedetniemi, Petra Hoffmannová, Svante Janson, Ida Kantor, Mark Kayll, Don Knuth, János Komlós, Laci Lovász, Tomasz Łuczak, János Pach, Balázs Patkós, Staszek Radziszowski, Bruce Reed, Charles Reiss, Peter Renz, Vojta Rödl, Bruce Rothschild, Lex Schrijver, Gábor Simonyi, Benny Sudakov, Michael Tarsi, Zsolt Tuza, Avi Wigderson, Pete Winkler, and Yori Zwols sent me comments on preliminary versions of the manuscript, which helped me in improving them.

The enthusiasm that Katie Leach expressed for an early version of the manuscript has meant much to me. In her capable care as Cambridge Editor, I felt nurtured and protected.

I am grateful to all of these people.