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978-1-108-82159-9 — Lectures on Orthogonal Polynomials and Special Functions

Edited by Howard S. Cohl , Mourad E. H. Ismail

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