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Lectures on Orthogonal Polynomials and Special Functions

Sixth Summer School, Maryland, 2016

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Preface

On July 11–15, 2016, we organized a summer school, *Orthogonal Polynomials and Special Functions*, Summer School 6 (OPSF-S6) which was hosted at the University of Maryland, College Park, Maryland. This summer school was co-organized with Kasso Okoudjou, Professor and Associate Chair, Department of Mathematics, and Norbert Wiener Center for Harmonic Analysis and Applications. OPSF-S6 was a National Science Foundation (NSF) supported summer school on orthogonal polynomials and special functions which received partial support from the Institute for Mathematics and its Applications (IMA), Minneapolis, Minnesota. Twenty-two undergraduates, graduate students, and young researchers attended the summer school from the USA, China, Europe, Morocco and Tunisia, hoping to learn a new state of the art in these subject areas.

Since 1970, the subjects of *special functions* and special families of *orthogonal polynomials*, have gone through major developments. The Wilson and Askey–Wilson polynomials paved the way for a better understanding of the theory of hypergeometric and basic hypergeometric series and shed new light on the pioneering work of Rogers and Ramanujan. This was combined with advances in the applications of q-series in number theory through the theory of partitions and allied subjects. When quantum groups arrived, the functions which appeared in their representation theory turned out to be the same q-functions which were recently developed at that time. This motivated researchers to revisit the old Bochner problem of studying polynomial solutions to second order differential, difference, or q-difference equations, which are of Sturm–Liouville type and have polynomial coefficients. This led to a generalization where having a solution of degree n for all $n = 0,1,2,\ldots$ is now weakened to having a solution of degree n for all $n = N, N+1,\ldots$, for some finite N. The polynomial solutions are required to be orthogonal. This gave birth to the subject of exceptional orthogonal polynomials. What is amazing is that these orthogonal polynomials continued to be complete in the L_2 space weighted by their orthogonality measure. At the same time, harmonic analysis of the newly developed functions attracted the attention of many mathematicians with different backgrounds. Alongside the analytic theory of the newly discovered functions and polynomials, a powerful combinatorial theory of orthogonal polynomials evolved. This also included a combinatorial theory of continued fractions. In the late 1990s, the elliptic gamma function was introduced and the theory of elliptic special functions followed. This area overlaps with mathematical physics because there are physical models whose solutions use elliptic special functions. The theory of multivariate orthogonal polynomials and special functions, as well as the harmonic analysis of root systems was also developed. The Selberg integral from 1944 became a prime example of integrals over root systems.

In this summer school, we felt that we needed to cover topics of current interest, as well as newly evolving topics which are expected to evolve and blossom into very active areas of research. This is hard to do in a classical subject like special functions. We also tried to cover areas within the broad scope of this classical and well established subject which have not been covered in recent summer schools. The fruits of our labor are in the lecture notes written by the lecturers of the summer school contained in this volume. We believe the these notes are detailed enough to learn or teach graduate level classes from, or cover in advanced seminars. They are full of insights into the subjects covered, and are written by experts whose research is at the leading edge of the subject.

The lecturers were Antonio Durán (Departamento de Análisis Matemático, Universidad de Sevilla, Sevilla, Spain), Mourad Ismail (Department of Mathematics, University of Central Florida, Orlando, Florida, USA), Erik Koelink (Department of Mathematics, Radboud Universiteit Nijmegen, Nijmegen, The Netherlands), Hjalmar Rosengren (Chalmers University of Technology and University of Gothenburg, Gothenburg, Sweden), and Jiang Zeng (Institut Camille Jordan, Université Claude Bernard Lyon 1, Villeurbanne, Lyon, France).

The first set of these lecture notes is by Antonio Durán who has made significant contributions in the subject of matrix orthogonal polynomials and exceptional orthogonal polynomials. Both subjects are relatively new, but their roots go back to the 1940s and have evolved very rapidly since the 1990s. These notes give a very nice introduction to the subject of classical orthogonal polynomials, and covers some very recent results on two expansions of the Askey tableau: Krall and exceptional polynomials; both in their continuous and discrete versions. Critical to Durán's discussion is the theory of $\mathscr{D}$-operators, how they can be used to construct Krall polynomials, and the use of duality to construct exceptional discrete polynomials. Durán clearly explains the relations between all of these families via limiting processes and using Christoffel transformations of the measure (Darboux transformations). Examples are treated in detail including the orthogonal polynomial families of Krall–Laguerre, Krall–Meixner, Krall–Charlier, exceptional Hermite, exceptional Laguerre, exceptional Meixner, and exceptional Charlier polynomials. These notes discuss the above developments and describes as well the current state of the art.

Mourad Ismail's lectures represent a nice tutorial which gives a brief review of many useful and recent results in the theory of q-series. He uses the Askey–Wilson operators, polynomial expansions, and analytic continuation to derive many of the fundamental identities and transformations of q-series. The Askey–Wilson calculus plays a major role in his approach, and even though this approach was also used in his 2005 book, our understanding of these analytic and more conceptional techniques has clearly advanced in the last few years. Some of these techniques were already used for the development of the theory of hypergeometric functions on root systems. This approach is more conceptual than the classical approach presented in most books on the subject. We expect this tutorial to be an advantageous text for learners of the subject.

The lectures by Erik Koelink cover the spectral theory of self-adjoint operators related to, or arising from, problems involving special functions and orthogonal polynomials. Of paramount importance is the spectral decomposition (the spectral theorem) of compact bounded and unbounded self-adjoint operators on infinite-dimensional Hilbert spaces. In particular, the emphasis in these notes is on the spectral theory of Jacobi operators, and especially on Jacobi operators corresponding to matrix-valued orthogonal polynomials. The theory of such Jacobi operators is built up, and several explicit examples are presented. In addition, he surveys the rigorous mathematical treatment of the J-matrix method and the more general concept of tridiagonalization. The J-matrix method was originally introduced by physicists and has been applied very successfully to many quantum mechanical systems.

Hjalmar Rosengren's lectures provides an excellent introduction to elliptic hypergeometric functions, a relatively recent class of special functions. These lecture notes fill a hole in the demand for a clear introduction to the field. He dedicates the first part of his lecture to a review of important topics connected to the subject, such as additive and multiplicative elliptic functions, theta functions, factorization, Weierstrass's three-term identity, even elliptic functions, partial fraction decomposition, modularity and elliptic curves. Throughout his lectures, through examples and a host of carefully constructed exercises, he connects his discussion with important and illustrative facts in complex function theory and to generalized and basic hypergeometric functions. He closes the lectures with a highly relevant historical application from statistical mechanics which is how elliptic hypergeometric series first appeared in the literature. This application concerns fused Boltzmann weights for Baxter's solvable elliptic solid-on-solid lattice model.

Note that in the abstract of his lectures, Rosengren says "By contrast, there is no hint about elliptic hypergeometric series in the literature until 1988." This statement is definitely true, but Mourad Ismail would like to shed some light on this issue. In 1982, he visited the Chudnovsky brothers and Gregory Chudnovsky asked him to explore orthogonal polynomials whose three-term recurrence coefficients are elliptic functions. Gregory also said that an elliptic generalization of the continuous q-ultraspherical polynomials should exist. Mourad was unable to solve this problem, but Gregory Chudnovsky anticipated the *elliptic revolution* in hypergeometric functions.

It is our opinion that a mathematical theory is best understood when formulated at the correct level of generality. This is indeed the case with the one variable theory of special functions. It is conceptually simpler at the most general level, the elliptic level.

The lecture notes by Jiang Zeng report on some recent topics developed in the cross-cutting field of combinatorics, special functions and orthogonal polynomials. This subject is relatively recent, although it is rooted in earlier works from the 1940s by Touchard, Kaplansky, Riordan, and Erdös. Zeng presents side-by-side, the two dual combinatorial approaches to orthogonal polynomials, namely that of Flajolet and Viennot, and that of Foata.

The lectures begin by reviewing essential results in the theory of general orthogonal polynomials, namely Favard's theorem, three-term recurrence relations and their connection to continued fraction expansions through generating functions for the moments. Since, orthogonal polynomials can be considered as enumerators of finite structures with respect to some statistics, Zeng illustrates how to use combinatorial models to explore orthogonal polynomial identities. As examples, Zeng develops combinatorial models for orthogonal Sheffer polynomials (Hermite, Laguerre, Charlier, Meixner, and Meixner–Pollaczek) through exponential generating functions, and for Al-Salam–Chihara polynomials (and their rescaled special sub-families including continuous q-Hermite, q-Charlier, and q-Laguerre polynomials) to combinatorially compute their moments and linearization coefficients. Zeng gives some details about Ismail, Stanton and Viennot's groundbreaking combinatorial evaluation of the Askey–Wilson integral and inquires whether there may exist combinatorial proofs of the Nassrallah–Rahman integral and Askey–Wilson moments. There are even recent applications of this approach to the asymmetric simple exclusion process (ASEP), where the moments of the Askey–Wilson polynomials play a central role.

Organizing such a summer school was a pleasure, although we encountered many challenges along the way. Kasso Okoudjou led the local organizational efforts for OPSF-S6. It was a pleasure working with Kasso throughout the entire process, both before and after the summer school. He skillfully maneuvered all aspects of the summer school, including the successful procurement of NSF and IMA funding, without which OPSF-S6 would not have been nearly as successful as it was. Kasso smoothly interacted with graduate students, early career researchers, and lecturers, for travel planning, booking, and ultimately for reimbursement. It was both an honor and a pleasure working with Kasso whose influence and organization with the Norbert Wiener Center for Harmonic Analysis and Applications is greatly appreciated. The lecture rooms we used, that Kasso obtained, were both useful and comfortable; snacks and coffee were always available; and the local staff was extremely pleasant and efficient. We would also like to acknowledge the support of Dan Lozier and Stephen Casey for their help and advice in planning the summer school.

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