New Handbook of Mathematical Psychology

Volume 3: Perceptual and Cognitive Processes

The field of mathematical psychology began in the 1950s and includes both psychological theorizing, in which mathematics plays a key role, and applied mathematics motivated by substantive problems in psychology. Central to its success was the publication of the first *Handbook of Mathematical Psychology* in the 1960s. The psychological sciences have since expanded to include new areas of research, and significant advances have been made both in traditional psychological domains and in the applications of the computational sciences to psychology. Upholding the rigor of the original Handbook, the *New Handbook of Mathematical Psychology* reflects the current state of the field by exploring the mathematical and computational foundations of new developments over the last half-century. The third volume provides up-to-date, foundational chapters on early vision, psychophysics and scaling, multisensory integration, learning and memory, cognitive control, approximate Bayesian computation, and encoding models in neuroimaging.

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Volume 3. Perceptual and Cognitive Processes

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CAMBRIDGE
UNIVERSITY PRESS

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Shaftesbury Road, Cambridge CB2 8EA, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India

103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of Cambridge University Press & Assessment, a department of the University of Cambridge.

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www.cambridge.org
Information on this title: www.cambridge.org/9781108821223

DOI: 10.1017/9781108902724

First published 2023
First paperback edition 2024

A catalogue record for this publication is available from the British Library

ISBN 978-1-108-83067-6 Hardback
ISBN 978-1-108-82122-3 Paperback

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Preface

In 1845 Edgar Allan Poe published a story titled "The Purloined Letter," in which a protagonist, Mr. C. Auguste Dupin, says the following:

> The mathematics are the science of form and quantity; mathematical reasoning is merely logic applied to observation upon form and quantity. The great error lies in supposing that even the truths of what is called pure algebra, are abstract or general truths. And this error is so egregious that I am confounded at the universality with which it has been received. Mathematical axioms are not axioms of general truth. What is true of relation — of form and quantity — is often grossly false in regard to morals, for example. In this latter science it is very usually untrue that the aggregated parts are equal to the whole. [. . .] two motives, each of a given value, have not, necessarily, a value when united, equal to the sum of their values apart. There are numerous other mathematical truths which are only truths within the limits of relation. But the mathematician argues, from his finite truths, through habit, as if they were of an absolutely general applicability — as the world indeed imagines them to be.

A safe reaction to this excerpt (especially in view of Mr. Dupin's subsequent remarks, omitted here) is that Mr. Dupin has a hopelessly approximate notion of mathematics. However, his appellation to morals and motives provides an opportunity for a more generous reaction, making Mr. Dupin's tirade relevant to a discussion of mathematical psychology. One could interpret this tirade as stating that

- D1 given two motives or moral ideas A and B that are combined in some well-defined sense (e.g., co-occur chronologically),
- D2 and assuming that each of them can be assigned a value represented by a real number, a and b,
- D3 and assuming that the combination of A and B can also be assigned a value c that is a real number,
- D4 and assuming that the combination of A and B is represented by the sum of their individual values, $a + b$,
- D5 we observe empirically that the value c is not generally equal to $a + b$;
- D6 the contradiction between D4 and D5 shows that the laws of arithmetic do not apply to motives and moral ideas.

Of course, the assumptions D1–D4 are hidden, they are not explicated by Mr. Dupin. Nor would he stop to think about how he could know the truth of D5. Deny any of the assumptions D1–D5, and Mr. Dupin will lose any grounds to blame mathematics. For instance, if the assumption D4 is not made, then c does not have to be equal to $a + b$, it can instead be ab or $\max(a, b)$, or perhaps a and b alone do not determine c at all. Mathematics is perfectly fine with these possibilities. Mathematics is also fine with the possibility that the assumptions D2 and D3 are wrong, and the motives or moral ideas are not representable by anything that can be subjected to conventional addition. Perhaps a and b are dimensioned numbers, but their dimensionality is not the same (say, they are measured in "love units" and "revenge units," respectively).

Is there any useful lesson that can be derived from this admittedly too easy critique of Mr. Dupin's perorations? We think there is. The lesson is that mathematics in psychology (or chemistry, or wherever else it is applied) is not about adding, multiplying, or, generally, computing. It is primarily about striving for conceptual clarity and avoiding conceptual confusions. Before we can compute, we need to explicate the hidden assumptions we make, and often when we do this we find out these assumptions are not all that compelling.

Take as an example the following piece of reasoning one can encounter in the modern literature. In logic, the conjunction of two statements is commutative, $A\&B$ is the same as $B\&A$. However, we have empirical evidence that the chronological order in which two statements are presented or evaluated does matter for one's judgment of the truth value (or probability) of their conjunction. Ergo, classical logic (probability theory) is not applicable to human judgments. Let us see what is involved in this reasoning.

- L1 Assuming that if A is presented first and B is presented second, then their combination is represented by $A\&B$,
- L2 whence, by symmetry, if A is presented second and B is presented first, their combination is represented by $B\&A$;
- L3 and knowing from classical logic that $A\&B$ and $B\&A$ are equivalent,
- L4 their truth value (or probability) M should be the same, $M(A\&B) = M(B\&A)$.
- L5 But empirical observations tell us this is not generally the case.
- L6 Ergo, classical logic (classical probability) here does not work.

The reasoning here is definitely "Dupinesque." Far from not being applicable, formal logic, if applied correctly, would lead one to reject, by *reductio ad absurdum*, the assumptions L1 and L2. Indeed, L3 and the implication L3→L4 are unassailable, and we assume L5 truthfully describes empirical facts. The ways to constructively deny L1 and L2 readily suggest themselves. One way is to introduce a special, noncommutative operation *A then B*. Another way is to identify the statements not only by their content but also by their chronological position in the pair: a statement with content A, if presented first, is A_1, when presented second it is A_2. So the rejected representations $A\&B$ and $B\&A$ in L1 and L2 are in reality $A_1\&B_2$ and $A_2\&B_1$, respectively. The commutativity of the conjunction is perfectly preserved, e.g., $A_1\&B_2 \equiv B_2\&A_1$. But $A_1\&B_2$ and $A_2\&B_1$ are different propositions, and one should generally expect that

$$M\left(A_1\&B_2\right) \neq M\left(A_2\&B_1\right),$$

whatever M may be. One can further investigate which of the two solutions, the introduction of *A then B* or the positional labeling, is preferable. Thus, if the truth values of the statements A and B themselves, and not just of their conjunction, depends on their chronological position, then the positional labeling clearly wins.

The quest for conceptual clarity and explication of hidden assumptions often faces greater and subtler difficulties than in the examples above. The greater then are the rewards ensuing from resolving these difficulties. Take as an illustration the question of whether the ways we measure certain quantities constrain the way these quantities can be related to each other. The historical context for this question is the emergence in mathematical psychology in the second half of the twentieth century of the line of research referred to as representational theory of measurement. It is an unusual theory, in the sense that while it is a firmly established branch or part of mathematical psychology, its aim is to formalize all empirical measurements, across sciences, and even provide necessary conditions for all possible scientific laws and regularities.

One of the tenets of this theory, widely accepted in modern psychology (and in textbooks of elementary statistics), is that all entities we deal with, physical or mental, are measured on specific scales, such as ordinal, interval, or ratio scales. We need not get here into the details of the qualitative, or pre-numerical, symmetries (automorphisms) postulated for the entities being measured. Suffice it to mention that the scale type assigned to these entities is characterized by the class (usually, a parametric group) of interchangeable mathematical representations, i.e., measurement functions, mapping the entities being measured into mathematical objects, usually real numbers. Thus, if entities $\mathbf{x} \in \mathcal{X}$, say, stimulus intensity or sensation magnitude values, are said to be measured on a ratio scale, it means that the measurement functions for $\mathcal{X}$ map this set into intervals of real numbers, and that if f and g are such measurement functions, then, for every $\mathbf{x} \in \mathcal{X}$,

$$f\left(\mathbf{x}\right) = kg\left(\mathbf{x}\right),$$

for some positive constant k. R. Duncan Luce, arguably one of the two greatest mathematical psychologists of the twentieth century (along with William K. Estes), made use of the notion of a measurement scale to restrict theoretically the class of possible psychophysical functions, those relating the magnitude of stimulus to the magnitude of sensation it causes. Luce proposed this idea in a book entitled *Individual choice behavior* (Luce, 1959a) and in a journal article (Luce, 1959b). The idea is so attractive aesthetically that it deserves being reproduced here, *mutatis mutandis*.

Let $x = f\left(\mathbf{x}\right)$ and $s = \varphi\left(\mathbf{s}\right)$ represent measurement functions for the stimulus magnitude $\mathbf{x}$ and sensation magnitude $\mathbf{s}$, respectively, and let the psychophysical function relating **s** to **x**, written in terms of these specific measurement functions, be

$$s = \psi\left(x\right).$$

Assume that both **x** and **s** are of the ratio-scale type. Consider another admissible measurement function for **x**:

$$x' = kf\left(\mathbf{x}\right),$$

for some $k > 0$. Then, Luce hypothesized, if one switches from x to x', the psychophysical function should be presentable as

$$s' = \psi\left(x'\right),$$

where

$$s' = c\varphi\left(\mathbf{s}\right),$$

for some $c > 0$. That is, s' is another admissible measurement function for **s**. Put differently, the function ψ is invariant with respect to admissible changes of the measurement function for **x**, provided that the measurement function for the dependent variable **s** can also change to other measurement functions accordingly. The last word, "accordingly," means that the choice of the measurement function for **s** generally depends on the choice of the measurement function for **x**, i.e.,

$$c = K\left(k\right),$$

for some function K.

The reasoning here is seductively plausible, and Luce thought that examples of the well-established laws of physics confirmed its validity. Thus, Newton's law of gravitation is conventionally written as

$$F = \gamma\frac{m_1 m_2}{r^2}.$$

If we assume that everything on the right-hand side is fixed except for the distance measurement function r, then augmenting this measurement function by the factor of $k = 10$ would result in the same expression, except that the measurement function F for force will have to be multiplied by $c = k^{-2} = 1/100$.

Having accepted Luce's hypothesis (Luce called it a "principle of theory construction"), we are led to a surprising conclusion: the psychophysical function cannot be anything but a power function. What is surprising here is that this conclusion is based on no empirical evidence, it is obtained deductively, by merely assuming that the magnitudes of stimulus and sensation are of the ratio-scale type. Indeed, the reasoning above translates into

$$\psi\left(kx\right) = \psi\left(x'\right) = s' = cs = c\psi\left(x\right) = K\left(k\right)\psi\left(x\right),$$

whence, by eliminating all but the marginal terms, we get the functional equation

$$\psi\left(kx\right) = K\left(k\right)\psi\left(x\right).$$

Here, the values of x and k are positive, and the functions ψ and K are positive and increasing. Since the functional equation holds for all positive k and all x on some interval of positive reals, its only solution is known to be (Aczel, 1987)

$$\psi(x) = bx^{\beta}, K(k) = k^{\beta},$$

for some positive b and β.

It looks like we have here an immaculate piece of deductive reasoning, with all concepts rigorously defined and all assumptions explicated. However, what shall we do with the fact that psychophysical laws of other forms have been proposed too? Most notably, every psychologist knows of the logarithmic law proposed by Gustav Theodor Fechner in 1861:

$$s = s_0 \log \frac{x}{x_0}.$$

Here, x_0 is the numerical representation of the absolute threshold magnitude $\mathbf{x_0}$, one at which the numerical representation of $\mathbf{s}$ is zero, for all measurement functions.

We can see that Fechner's law does not violate any of Luce's assumptions. Since $\mathbf{x}$ and $\mathbf{x_0}$ are measured by the same measurement function, the value of

$$\frac{f(\mathbf{x})}{f(\mathbf{x_0})} = \frac{kx}{kx_0}$$

is the same for all admissible f. The magnitude of the absolute lower threshold is defined irrespective of the measurement function chosen for $\mathbf{x}$, because so is defined $s = 0$. Even if one denies the existence of absolute threshold as a fixed constant, such operational definitions of $\mathbf{x_0}$ as "the value of $\mathbf{x}$ detected with probability p" are independent of the measurement function for $\mathbf{x}$. The measurement function for the dependent variable $\mathbf{s}$ is chosen independently, which formally translates into $K(k) = 1$. The value s_0 is the numerical representation of the value of $\mathbf{s}$ corresponding to the value of $\mathbf{x}$ at which $\log \frac{x}{x_0} = 1$.

Since the logarithmic law is not the same as the power law, Luce must have made a hidden assumption that Fechner's derivation of his law violates. This hidden assumption is not difficult to detect. It is the assumption that the dependence of $\mathbf{s}$ on $\mathbf{x} \in \mathcal{X}$ contains no parameters (constants with respect to $\mathbf{x}$) that belong to the same set $\mathcal{X}$ and are therefore represented by the same measurement function. Such parameters are called measurement-dependent constants, or dimensional constants in the case of ratio scales. An expression

$$s = s_0 \psi \left(\frac{x}{x_0} \right),$$

with dimensional constants x_0 and s_0, can hold for any positive increasing function ψ. Using examples of physical laws, this was pointed out to Duncan Luce by William W. Rozeboom in a 1962 article (Rozeboom, 1962). Being a true scientist, Luce accepted this criticism and withdrew his "principle of theory construction" (Luce, 1962). Interestingly, in the formulation of this principle, Luce did in fact mention dimensional constants: the form of the dependence ψ should be invariant, he wrote, "except for the numerical values of parameters that reflect the effect on the dependent variables of admissible transformations of the independent variables." This is precisely what dimensional constants are. Using Luce's own example of the universal gravitation law, in the formula

$$F = \gamma \frac{m_1 m_2}{r^2},$$

if one uses the distance–time–mass–force system of units, changing the dimensionality of mass or distance in no way leads to the change of the dimensionality of force. Rather, the dimensional constant γ, whose dimensionality is

$$force \cdot distance^2 \cdot mass^{-2},$$

changes its numerical value. In essence, γ is a coalesced form (using the expression coined by Percy Williams Bridgman) of the "individual" dimensional constants in the formula

$$\frac{F}{F_0} = \frac{\frac{m_1}{m_0}\frac{m_2}{m_0}}{\left(\frac{r}{r_0}\right)^2}.$$

The lesson we learn from the story of Duncan Luce's "principle of theory construction" is that hidden assumptions and lack of conceptual clarity due to the failure to explicate them can be present even in very rigorous treatments. Moreover, explication of these hidden assumptions, while resolving the issue at hand, leads to new conceptual problems and opens new avenues of conceptual research. In our example the new conceptual problems can be formulated thus:

P1 What is the nature of dimensional (more generally, measurement-dependent) constants in empirical laws? Where do they come from?

P2 How do we know the scale type (the group of admissible measurement functions) of a given entity? Is it imposed on the entity by the human mind, or is it objectively present in it, to be uncovered?

These questions are at the foundations of all empirical science, and it is an interesting historical fact that their development owes a great deal to mathematical psychology (see, e.g., Dzhafarov, 1995; Falmagne & Doble, 2018; Narens, 2007). This preface, of course, is not a place to discuss these questions in any detail.

About this Volume

This is the third, and concluding, volume of the *New Handbook of Mathematical Psychology*. In the same way as the first two volumes, it offers a representative sample of several branches of mathematical psychology. This volume focuses on sensory and perceptual processing, learning and memory, and cognition.

Chapter 1, written by Brian Wandell and David Brainard, surveys low-level encoding of visual information. Modern vision science is highly interdisciplinary, combining ideas from physics, biology, and psychology. In recent years, deductive mathematics in vision science is often combined with computational modeling to add realism to the mathematical formulations. Together, the mathematics and computational tools provide a realistic estimate of the initial signals that the brain analyzes to render visual judgments of various aspects of visual image, such as motion, depth, and color. The chapter first traces the calculations from the representation of the light signal, to how that signal is transformed by the lens to the retinal image, and then how the image is converted into the cone photoreceptor excitations. The central steps in the initial encoding rely heavily on linear systems theory and the mathematics of signal-dependent noise. The chapter describes computational methods used to understand how light is encoded by cone excitations. The chapter also provides a mathematical formulation of the ideal observer that uses all the encoded information to perform a visual discrimination task, as well as Bayesian methods that combine prior information and sensory data to estimate the light input. These tools help one to reason about what information is present in the neural representation, what information is lost, and what types of neural circuits could extract information to make judgments about a visual scene.

Chapter 2, by Adele Diederich and Hans Colonius, deals with the topic of multisensory integration – that is, with the merging of the information provided by different sensory modalities. This topic has been the subject of many competing theories, often crossing boundaries between psychology and neuroscience. In defining the somewhat fuzzy term of "multisensory integration," it has been observed that at least some kind of numerical measurement assessing the strength of the crossmodal effects is always required. The focus of this chapter is on measures of multisensory integration based on both behavioral and single-neuron recording data: spike numbers, reaction time, frequency of correct or incorrect responses in detection, recognition, and discrimination tasks. On the empirical side, these measures typically serve to quantify effects on multisensory integration of attention, learning, and such factors as age, certain disorders, developmental conditions, training and rehabilitation. On the theoretical side, these measures often help to quantify important characteristics of multisensory integration, such as optimality in combining information or inverse effectiveness, without necessarily subscribing to any specific model of the mechanisms of multisensory integration.

Ehtibar Dzhafarov and Hans Colonius present a systematic theory of generalized (or universal) Fechnerian scaling in Chapter 3 that is based on the intuition underlying Fechner's original theory. This intuition is that subjective distances among stimuli are computed by means of cumulating small discriminability values between "neighboring" stimuli. A stimulus space is supposed to be endowed by a dissimilarity function, computed from a discrimination probability function for any pair of stimuli chosen in two distinct observation areas. On the most abstract level, one considers all possible chains of stimuli leading from a stimulus **a** to a stimulus **b** and back to **a**, and takes the infimum of the sums of the dissimilarities along these chains to be the subjective distance between **a** and **b**. In arc-connected spaces, the cumulation of dissimilarity values along all possible chains reduces to their cumulation along continuous paths, leading one to a fully fledged metric geometry. In topologically Euclidean spaces, the cumulation along paths further reduces to integration along smooth paths, and the geometry in question acquires the form of a generalized Finsler geometry. The chapter also discusses such related issues as Fechner's original derivation of his logarithmic law, an observational version of the sorites paradox, a generalized Floyd–Warshall algorithm for computing metric distances from dissimilarities, an ultra-metric version of Fechnerian scaling, and data-analytic applications of Fechnerian scaling.

Gregory Ashby, Matthew J. Crossley, and Jeffrey Inglis review mathematical models of human learning in Chapter 4. Although learning was a key focus during the early years of mathematical psychology, the cognitive revolution of the 1960s caused the field to languish for several decades. Two breakthroughs in neuroscience resurrected the field. The first was the discovery of long-term potentiation and long-term depression, which served as promising models of learning at the cellular level. The second was the discovery that humans have multiple learning and memory systems that each require a qualitatively different kind of model. Currently, the field is well represented at all of Marr's three levels of analysis. Descriptive and process models of human learning are dominated by two different but converging approaches – one rooted in Bayesian statistics and one based on popular machine-learning algorithms. Implementational models are in the form of neural networks that mimic known neuroanatomy and account for learning via biologically plausible models of synaptic plasticity. Models of all these types are reviewed, and advantages and disadvantages of the different approaches are considered.

Marc W. Howard's Chapter 5 surveys formal models of memory. The idea that memory behavior relies on a gradually changing internal state has a long history in mathematical psychology. The chapter traces this line of thought from statistical learning theory in the 1950s, through distributed memory models in the latter part of the twentieth century and early part of the twenty-first century, through to modern models based on a scale-invariant temporal history. The author discusses the neural phenomena consistent with this form of representation and sketches the kinds of cognitive models that can be constructed with its use, in connection with formal models of various memory tasks.

In Chapter 6, Gregory Ashby and Michael Wenger review statistical decision theory, which provides a general account of perceptual decision-making in a wide variety of tasks that range from simple target detection to complete identification. The fundamental assumptions are that all sensory representations are inherently noisy and that every behavior, no matter how trivial, requires a decision. Statistical decision theory is referred to as signal detection theory (SDT) when the stimuli vary on only one sensory dimension, and as general recognition theory (GRT) when the stimuli vary on two or more sensory dimensions. SDT and GRT are both reviewed. The SDT review focuses on applications to the two-stimulus identification task and multiple-look experiments, and on response-time extensions of the model (e.g., the drift-diffusion model). The GRT review focuses on applications to identification and categorization experiments, and in the former case, especially on experiments in which the stimuli are constructed by factorially combining several levels of two stimulus dimensions. The basic GRT properties of perceptual separability, decisional separability, perceptual independence, and holism are described. In the case of identification experiments, the summary statistics methods for testing perceptual interactions are described, and so is the model-fitting approach. Response time and neuroscience extensions of GRT are reviewed.

Chapter 7, written by Hans Colonius and Adele Diederich, deals with response inhibition, which is an organism's ability to suppress unwanted impulses, or actions and responses that are no longer required or have become inappropriate. In a stop-signal task experiment, participants perform a response time task (go task), and occasionally the go stimulus is followed by a stop signal after a variable delay, indicating subjects to withhold their response (stop task). The main interest of modeling is in estimating the unobservable latency of the stopping process as a characterization of the response inhibition mechanism. The authors analyze and compare the underlying assumptions of different models, including parametric and nonparametric versions of the race model. New model classes based on the concept of copulas are introduced, and a number of unsolved problems facing all existing models are pointed out.

In Chapter 8, written by Noah Thomas, Brandon M. Turner, and Trisha Van Zandt, approximate Bayesian analysis is presented as the solution for complex computational models where no explicit maximum likelihood estimation is possible. The activation-suppression race model (ASR), which does have a likelihood amenable to Markov chain Monte Carlo methods, is used to demonstrate the accuracy with which parameters can be estimated with the approximate Bayesian methods.

The cognitive diagnosis models considered in Chapter 9 by Jimmy de la Torre and Miguel A. Sorrel have their historical origins in the field of educational measurement, as a psychometric tool to provide finer-grained information suitable for formative assessment. Typically, but not necessarily, these models classify examinees as masters and nonmasters on a set of binary attributes. The chapter aims to provide a general overview of the original models and the extensions, and methodological developments, that have been made in the last decade. The topics covered in the chapter include model estimation, Q-matrix specification, model fit evaluation, and procedures for gathering validity and reliability evidences. The chapter ends with a discussion of future trends in the field.

Finally, Chapter 10, written by Fabian Soto and Gregory Ashby, reviews encoding models in neuroimaging. This is the neuroimaging area closest to mathematical psychology in which models of neuroimaging data are constructed by combining assumptions about underlying neural processes with knowledge of the task and the type of neuroimaging technique being used to produce equations that predict values of the dependent variable that is measured at each recording site (e.g., the fMRI BOLD response). Voxel-based encoding models include an encoding model that predicts how every hypothesized neural population responds to each stimulus, and a measurement model that first transforms neural population responses into aggregate neural activity and then into values of the dependent variable being measured. Encoding models can be inverted to produce decoding schemes that use the observed data to make predictions about what stimulus was presented on each trial, thereby allowing unique tests of a mathematical model. Representational similarity analysis is a multivariate method that provides unique tests of a model by comparing its predicted similarity structures to similarity structures extracted from neuroimaging data. Model-based fMRI is a set of methods that were developed to test the validity of purely behavioral computational models against fMRI data. Collectively, encoding methods provide useful and powerful new tests of models – even purely cognitive models – that would have been considered fantasy just a few decades ago.

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