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Excerpt
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Introduction

The Australian–German Workshop on Differential Geometry in the Large was a two week programme that ran from the 4th of February 2019 through to the 15th of February 2019, under the auspices of the Mathematical Research Institute (MATRIX), at the old Victorian School of Forestry, in the town of Creswick, Australia, that operates now as a campus of the University of Melbourne.

The first week, 4th to the 8th, was a large international conference with a list of prominent keynote speakers from across the globe. These included Ben Andrews and Neil Trudinger from Australia, Rod Gover from New Zealand, Christoph Böhm and Burkhard Wilking from Germany, Robert Bryant, Karsten Grove, Claude LeBrun, Peter Petersen and Guofang Wei from the United States, Dame Frances Kirwan from the United Kingdom and Tom Farrell and Fuquan Fang from the People's Republic of China. In addition to these keynote speakers a breadth of contributed talks were delivered across a full gamut of topics in Differential Geometry and Geometric Analysis.

In the second week, 11th through to the 15th, the meeting became a research symposium with smaller specialist sessions with the themes that included:

- Geometric evolutions equations and curvature flow,
- Structures on manifolds and mathematical physics,
- Recent developments in non-negative sectional curvature.

In the afternoons, small teams of academics teamed up to work on existing or new projects. Some of the work in this volume was conceived at the meeting.

Part I. Geometric Evolution Equations and Curvature Flow

Geometric flows, particularly the Ricci flow and mean curvature flow, have been of great interest in recent decades.

Given a Riemannian manifold, (M, g), one defines the Ricci flow as the family of Riemannian metrics, (M, g_t), that solve the initial value problem,

$$\frac{\partial g_t}{\partial t} = -2\mathrm{Ric}_{g_t},$$
$$g_0 = g,$$

where Ric_{g_t} represents the Ricci curvature of the metric, g_t.

The Ricci flow since its introduction by Hamilton in 1981 has played an important rôle in the solution of several long standing conjectures. Along with Alexandrov geometry (another topic discussed at the meeting), the Ricci flow played a crucial rôle in Perel'man's resolution of Thurston's geometrization conjecture with the celebrated Poincaré conjecture as a corollary. Other interesting applications of the Ricci flow include Schoen and Brendle's solution of the quarter pinched differentiable sphere theorem and Böhm and Wilking's solution to the Hamilton conjecture that a Riemannian manifold with positive curvature operator is diffeomorphic to a positively curved space form. The study of Ricci flow has bearing on the existence of Ricci solitons and Kähler–Einstein metrics.

The mean curvature flow and its generalizations enjoy interesting smoothing properties for families of submanifolds.

Given a Riemannian embedding, $x : X^{n-1} \hookrightarrow M^n$, the mean curvature flow is a family of immersions, $x_t : X^{n-1} \to M^n$, solving the initial value problem,

$$\frac{\partial x_t}{\partial t} = -H_{x_t}\hat{n},$$
$$x_0 = x,$$

where H_{x_t} is the mean curvature of the hypersurface, $X_t = x_t(X)$, and $\hat{n}$ is the normal of the hypersurface, X_t. The mean curvature flow has minimal surfaces as its critical points and is useful in solving the isoperimetric problem.

Part II. Structures on Manifolds and Mathematical Physics

Structures on Riemannian manifolds can take a variety of forms either through the existence of vector fields or tensor fields or curvature properties. For instance, a Riemannian manifold, (M, g), is said to be Kähler if it carries a parallel almost complex structure, J, i.e. an almost complex structure such that $\nabla J = 0$. A Riemannian manifold, (M, g), is said to be conformally Kähler if there is a conformal factor, f, such that the Riemannian manifold, (M, fg), is Kähler. A Riemannian manifold, (S, g), is said to be Sasakian if it carries a unit Killing, ξ, such that

$$R(X, \xi)Y = g(Y, \xi)X - g(X, Y)\xi$$

for any pair of smooth vector fields, X, Y. Equivalently, (S, g) is Sasakian if the Riemannian cone, $(S \times (0, \infty), t^2 g + dt^2)$, is Kähler.

As an example of a structure on a Riemannian manifold defined in terms of a curvature property consider that a Riemannian manifold, (M, g), is said to be Einstein if there is a constant, λ, such that

$$\mathrm{Ric}_g = \lambda g.$$

One may ask if a given manifold carries a metric with a particular curvature property such as an Einstein metric, or more generally, given a symmetric 2 tensor, T, whether there exists a metric, g, with

$$\mathrm{Ric}_g = T,$$

the so called prescibed Ricci curvature problem.

Another problem concerning the existence of metrics with positive scalar curvature was posed by Hidehiko Yamabe in 1960.

Yamabe problem Given a Riemannian metric, (M, g), where N is compact of dimension 3 or greater, is there a smooth function, $f : M \to \mathbb{R}$, such that the $e^{2f}g$ has constant scalar curvature?

The answer to this problem is yes, though the demonstration proved somewhat more difficult and labour intensive than Yamabe had anticipated, fruitfully occupying Trudinger, Schoen and Aubin for years hence.

More obscure perhaps is the Obata problem.

Obata problem Given the standard Riemannian metric on the sphere, (S^d, g), is there a smooth function, $\sigma : S^d \to \mathbb{R}$, such that

$$|d\sigma|^2 - \frac{2\sigma}{d}\left(\Delta\sigma + \frac{\sigma}{2(d-1)}s\right) = 1$$

where $\sigma^{-2}g$ is not Einstein.

Structures on Riemannian manifolds, (M^n, g), are often closely related to constraints on its holonomy group (the group generated by parallel transport along curves in the manifold). A Riemannian manifold is said to possess *special holonomy* if its holonomy is a proper subgroup of the orthogonal group, $\mathrm{O}(n)$. For example, (M^n, g) is Kähler if and only if it is even dimensional and its holonomy is contained in $\mathrm{U}\left(\frac{n}{2}\right)$.

Metrics with special holonomy are rare and are expected to strongly constrain the topology of the manifold. Indeed, a celebrated theorem of Deligne and Sullivan states that Kähler manifolds have rational homotopy types which are formal. Through the work of Sullivan, the rational homotopy type of a space is equivalent to certain computable algebraic data and formality is a strong constraint on this data. More generally, it is conjectured that all manifolds with special holonomy will be formal. This conjecture is especially enticing in dimension 7, which is both the only dimension in which the exceptional Lie group, G_2, can appear as a holonomy group and the first dimension in which there are simply connected manifolds which are not formal. Despite concerted efforts, it is still unknown whether all manifolds with holonomy group G_2 are formal.

Part III. Recent Developments in Non-Negative Sectional Curvature

The study of manifolds of non-negative and positive sectional curvature has enjoyed some resurgence in the twenty first century. Although the study of sectional curvature is a topic that stretches back to the beginnings of Riemannian geometry, the study of positive sectional curvature in terms of a comprehensive search for examples only really began with Berger in 1960, with his classification of simply connected normal homogeneous spaces with positive curvature, and extended into the 1970s with Aloff and Wallach's discoveries and Berard-Bérgery's classification.

Methods of construction amount to the realization of the manifolds in question as orbit spaces of free isometric actions on Lie groups endowed with non-negatively curved metrics via the non-decreasing property of O'Neill's formula for the curvature of the target of a Riemannian submersion. As such the consideration of manifolds of non-negative curvature in a sense is a natural precursor to that of positive curvature. Indeed, Gromoll and Meyer's construction of a seven manifold homeomorphic, but not diffeomorphic to the standard 7-sphere as the orbit of an isometric action of $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ on $\mathrm{Sp}(2)$, obtained by multiplication on both the left and right (a so-called biquotient), is a stunning pioneering use of this simple technique. Eschenburg and Bazaikin went on to extend the technique to find new examples of manifolds with positive sectional curvature as biquotients in dimensions 7 and 13.

Developments in the twenty first century have considered cohomogeneity one actions on Riemannian manifolds and hinge on a key Lemma of Grove and Ziller that a cohomogeneity one manifold with codimension two singular orbits carries a metric with non-negative curvature. Essentially this amounts to the fact that as an orbit space the manifold can be obtained via the union of two non-negatively curved pieces that meet in isometric product collars. The examples of interest discussed by Grove and Ziller are the so-called Milnor spheres that occur as three sphere bundles over the 4-sphere. These arise in this context as isometric quotients of principal bundles that carry cohomogeneity one metrics of the above kind. Goette, Kerin and Shankar extended this technique more recently to apply to any exotic sphere in dimension 7. Some of the total spaces of the three sphere bundles also occur in the context of an up to now partially complete classification scheme of Grove, Wilking, Verdiani and Ziller of simply connected cohomogeneity one manifolds with positive sectional curvature, i.e. the so-called P_k family. This family, along with its companion family the Q_k, were also discussed at the meeting.

In both the study of non-negative and positive sectional curvature, as well as that of the more general theory of lower bounds on sectional, Ricci or scalar curvature, convergence techniques (notably ones involving the Gromov–Hausdorff distance) have for decades become an indispensable and ubiquitous tool in (and also way beyond) global differential geometry whenever it comes to studying sequences of metrics and their possible degenerations. This is, perhaps most prominently, illustrated by Gromov's proof of his celebrated theorem on groups of polynomial growth and Perel'man's proof of Thurston's geometrization conjecture.

Gromov–Hausdorff limits of Riemannian manifolds with a uniform lower bound on sectional curvature are known to be so-called Alexandrov spaces, that is, inner metric spaces of finite Hausdorff dimension with a metric notion of lower curvature bound satisfying variants of Toponogov's comparison theorem, and these interesting objects are nowadays studied in their own right as well. The length metric spaces that arise from limits of manifolds with lower bounds on Ricci curvature are so far much less understood. Their investigation constitutes today a very active field of research, involving, more generally, the study of RCD spaces, i.e. certain metric measure spaces which allow for synthetic notions of

Ricci curvature, based on methods of geometric measure theory, optimal transport and the foundational work of Lott–Villani and Sturm.

The meeting was a large and complex event that sponsored eight international keynote speakers' airfares as well as the airfares for many US based participants. The organizers wish to thank the Australian Mathematical Science Institute, the Ian Potter Foundation, the Australian Mathematical Society and La Trobe University for travel sponsorship of keynote speakers and the DFG national priority research scheme "Geometry at Infinity, SPP2026" and the National Science Foundation (NSF) for sponsoring the airfares of German and some US based participants, respectively; in particular, Lee Kennard deserves high praise for his work in successfully obtaining NSF travel funding. The organizers would also like to thank the International Conference Events Network in the New South Wales office of the Australian Department of Home Affairs for their quick work in securing a visa for one of our speakers giving a contributed talk in the first week.

The meeting provided full room and board to almost all participants as well as availing some paying places open to the general public, and the organizers wish to thank SPP2026, the NSF, MATRIX and the University of Melbourne International Research and Research Training Fund (IRRTF) for support with food and board for participants. The organizers would like to thank MATRIX for the provision of coaches from the old VSF to Melbourne Airport and Central Business District and the sponsorship for the provision of classrooms onsite. The organizers would like to thank the IRRTF for supporting the costs for refreshments, the welcoming dinner and farewell drinks at the Farmers' Arms Hotel and outings to Sovereign Hill and the Convent Gallery in Daylesford.

Owen Dearricott, Kyneton, March 2020, on behalf of the organizers,

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Group photograph of the attendees at the international conference week, 4th to 8th of February, 2019.

Group photograph of the attendees at the research symposium week, 11th to 15th of February, 2019.