Many Variations of Mahler Measures

The Mahler measure is a fascinating notion and an exciting topic in contemporary mathematics, interconnecting with subjects as diverse as number theory, analysis, arithmetic geometry, special functions and random walks. This friendly and concise introduction to the Mahler measure is a valuable resource for both graduate courses and self-study. It provides the reader with the necessary background material, before presenting recent achievements and remaining challenges in the field.

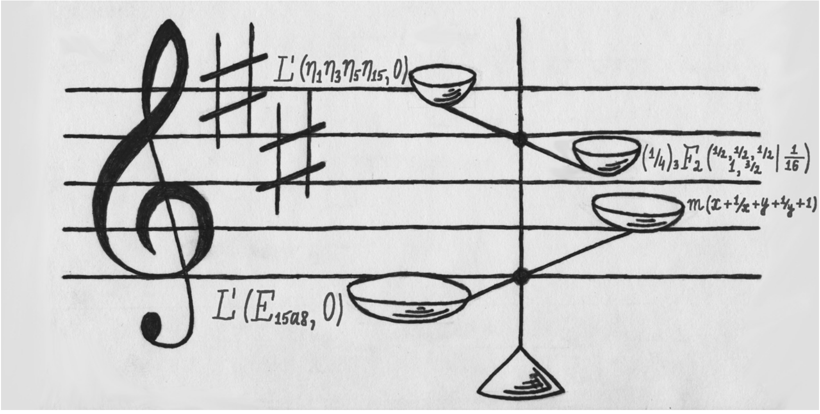
The first part introduces the univariate Mahler measure and addresses Lehmer's problem, and then discusses techniques of reducing multivariate measures to hypergeometric functions. The second part touches on the novelties of the subject, especially the relation with elliptic curves, modular forms and special values of L-functions. Finally, the appendix presents the modern definition of motivic cohomology and regulator maps, as well as Deligne–Beilinson cohomology. The text includes many exercises to test comprehension and to challenge readers of all abilities.

Cambridge University Press
978-1-108-79445-9 — Many Variations of Mahler Measures
François Brunault , Wadim Zudilin
Frontmatter
More Information

AUSTRALIAN MATHEMATICAL SOCIETY LECTURE SERIES

All the titles listed below can be obtained from good booksellers or from Cambridge University Press. For a complete series listing visit www.cambridge.org/mathematics.

8 Low Rank Representations and Graphs for Sporadic Groups, C. E. PRAEGER & L. H. SOICHER
9 Algebraic Groups and Lie Groups, G. I. LEHRER (ed.)
10 Modelling with Differential and Difference Equations, G. FULFORD, P. FORRESTER & A. JONES
11 Geometric Analysis and Lie Theory in Mathematics and Physics, A. L. CAREY & M. K. MURRAY (eds.)
12 Foundations of Convex Geometry, W. A. COPPEL
13 Introduction to the Analysis of Normed Linear Spaces, J. R. GILES
14 Integral: An Easy Approach after Kurzweil and Henstock, L. P. YEE & R. VYBORNY
15 Geometric Approaches to Differential Equations, P. J. VASSILIOU & I. G. LISLE (eds.)
16 Industrial Mathematics, G. R. FULFORD & P. BROADBRIDGE
17 A Course in Modern Analysis and its Applications, G. COHEN
18 Chaos: A Mathematical Introduction, J. BANKS, V. DRAGAN & A. JONES
19 Quantum Groups, R. STREET
20 Unitary Reflection Groups, G. I. LEHRER & D. E. TAYLOR
21 Lectures on Real Analysis, F. LÁRUSSON
22 Representations of Lie Algebras, A. HENDERSON
23 Neverending Fractions, J. BORWEIN et al.
24 Wavelets: A Student Guide, P. NICKOLAS
25 Classical Groups, Derangements and Primes, T. BURNESS & M. GIUDICI
26 Notes on Counting: An Introduction to Enumerative Combinatorics, P. J. CAMERON
27 Orthogonal Polynomials and Painlevé Equations, W. VAN ASSCHE


$L'(\eta_1\eta_3\eta_5\eta_{15}, 0)$
$(1/4)\,{}_3F_2\left(\begin{matrix}1/2, 1/2, 1/2\\ 1, 3/2\end{matrix}\,\middle|\,\frac{1}{16}\right)$
$m(x+1/x+y+1/y+1)$
$L'(E_{15a8}, 0)$

Australian Mathematical Society Lecture Series: 28

Many Variations of Mahler Measures

A Lasting Symphony

FRANÇOIS BRUNAULT
École Normale Supérieure de Lyon

WADIM ZUDILIN
Radboud Universiteit Nijmegen

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India

79 Anson Road, #06–04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781108794459
DOI:10.1017/9781108885553

First published 2020

Printed in the United Kingdom by TJ International Ltd, Padstow Cornwall

A catalogue record for this publication is available from the British Library.

ISBN 978-1-108-79445-9 Paperback

À mon père, qui a œuvré pour que
chaque personne se réalise pleinement

To Our Long-awaited Great Achievements

Contents

Preface

Kurt Mahler, Fellow of the Royal Society, is an illustrious part of Australia's mathematical heritage. His route to prominence, handicapped by tuberculosis in childhood, entailed completion of a PhD thesis under the great Carl Ludwig Siegel, shortly followed by his escape from Nazi Germany to England, and finally acceptance of a professorship at the newly opened Institute of Advanced Studies of the Australian National University in Canberra. At the beginning of the 1960s, at the very time of his move to Australia, Mahler [129–131] became interested in a particular height function of uni- and multivariate polynomials — a *measure*, its analytical representations and applications in number theory. In [130] a new version and a new proof of Gelfond's inequality relating the height of a product to the product of heights was given. This inequality is at the heart of one of the most successful methods in transcendental number theory, which led to the resolution of the seventh Hilbert problem independently by Gelfond and Schneider, and laid the groundwork for Alan Baker's method of linear forms in logarithms developed from the mid-1960s — Baker was awarded the Fields Medal in 1970 for this work. Years later, the original scope of the Mahler measure was expanded significantly after the discovery of its deep links to algebraic geometry and K-theory, in particular to Beilinson's conjectures. These interrelations generated a body of challenging problems, some recently resolved but many remaining open.

The set of notes and exercises below grew from numerous talks and lectures by the authors on Mahler measures, special values of L-functions, regulators and modular forms. This material was crystallised during the masterclass on the topic at the University of Copenhagen, Denmark, during the last week of August 2018. The set-up of two courses there was based on what are now Chapters 1–6 (lectured by W.Z.) and Chapters 7–10 (lectured by F.B.) of this book, with the lectures alternated. We take this opportunity to thank heartily

Fabien Pazuki and Riccardo Pengo, the organisers of this conference, as well as the participants for the inspiring and lively discussions around the lectures.

The book can still be viewed as consisting of two parts. The first part (Chapters 1–6) mainly touches the essential basics for the univariate Mahler measure and things related to Lehmer's problem (also known as Lehmer's question), then accelerates to multivariate settings and discusses techniques of reducing the measure to hypergeometric functions. Some L-functions (or L-values) related to (hyper)elliptic curves and modular forms show up in the background along the way. Then the discussion smoothly transitions into the second part of the book (Chapters 7–10), where the details of recent novelties of multivariable Mahler measures are given. Finally, the appendix presents the modern definition of motivic cohomology and regulator maps, as well as the example of Deligne–Beilinson cohomology.

The text is supplemented with many exercises of different complexity, and the chapters include additional ones that can help the reader to extend their horizon of understanding of the topic and its links to other mathematical gems. All this makes the book a unique and comprehensive introduction to a developing area, which has numerous links with practically any other part of mathematics and, at the same time, suffers from lack of systematic exposition. Apart from many references on the Mahler measure spread out in the literature (and cited in the bibliography list with care), we can point out the books [83, 176] and reviews [93, 198, 199] where certain aspects of the topic are covered.

This text is a friendly and concise introduction to the subject of Mahler measure, a potential source for a graduate course or self-study. There are definitely other aspects of Mahler measure to be discussed, but we have decided to stay realistic rather than encyclopaedic while going through a new — highly specialised! — subject. One reason behind the involvement of Mahler measure in so many areas of mathematics is that it bears the burden of a height function, the reason we reserve as an excuse for not covering all those numerous (and quite important) applications and links. The latter, for example, include applications to algebraic dynamics (well treated in [83]) and remarkable links to Szegő's limit theorems and integrable systems of statistical mechanics [178, 211].

During the writing process we have received valuable support and feedback from our friends, colleagues and collaborators, whom we would like to thank: Marie José Bertin, Denis-Charles Cisinski, Frédéric Déglise, Antonin Guilloux, Matilde Lalín, Hang Liu, Boaz Moerman, Michael Neururer, Fabien Pazuki, Riccardo Pengo, Berend Ringeling, Jörg Wildeshaus. Special thanks go to Michael Neururer for providing us with a `sage` code for the Riemann surface pictures, to Ralf Hemmecke and Silviu Radu for supplying us with

compact expressions for f_{21} and f_{45} in Chapter 6, as well as to Jörg Wildeshaus for allowing us to use the notes from his course at the summer school on special values of L-functions, organised at ÉNS Lyon in June 2014. We offer our heartfelt thanks to Roger Astley, Clare Dennison and Anna Scriven at Cambridge University Press, who provided expert support to bring the manuscript to publication. We are indebted to the copy-editor Alison Durham whose constructive feedback and professional work positively impacted the appearance of the book. The picture on the cover of this book was created by Victor Zudilin, whom we thank warmly for this creative and inspiring illustration of the topic of the book.

We hope that an allusion to the great composer Gustav Mahler in the subtitle of the book does not cause any confusion. Though the great mathematician Kurt Mahler seems to be not related to his famous namesake, his mathematics creature — the Mahler measure — is an artistic work comparable to a symphony in music. Enjoy its sounds!

François Brunault and Wadim Zudilin
Lyon, Nijmegen and Newcastle

Cambridge University Press
978-1-108-79445-9 — Many Variations of Mahler Measures
François Brunault , Wadim Zudilin
Frontmatter
More Information