1

Some basics

We first agree on customary terminology and notation for polynomials and algebraic numbers.

A polynomial $P(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0$ of degree d is said to be monic if its leading coefficient is $a_d = 1$. If all coefficients of $P(x)$ are integral, $P(x) \in \mathbb{Z}[x]$, and do not possess a common integral factor, $P(x)$ is said to be primitive. If $P(x) \in \mathbb{Z}[x]$ is not primitive or can be represented as the product of two polynomials from $\mathbb{Z}[x]$ of degrees less than $d = \deg P$, the polynomial $P(x)$ is called reducible; otherwise, it is irreducible (over $\mathbb{Z}$). Besides the degree, standard characteristics of polynomials are its length $L(P) = |a_d| + |a_{d-1}| + \cdots + |a_0|$ and its height $H(P) = \max_{0 \le j \le d} |a_j|$.

A (real or complex) number α is algebraic if it happens to be a zero of a non-zero polynomial with integral coefficients. To each algebraic number α one can assign its minimal polynomial $P(x)$ — a (primitive!) irreducible $P(x)$ such that $P(\alpha) = 0$; the polynomial is well defined up to sign. The degree, length and (naive) height of α are then defined as the degree, length and height, respectively, of its minimal polynomial $P(x)$. Furthermore, all zeros $\alpha_1 = \alpha$, $\alpha_2, \ldots, \alpha_d$ of the minimal polynomial $P(x)$ are called (algebraic) conjugates of α, so that $P(x) = a_d \prod_{j=1}^{d} (x - \alpha_j)$; in the case that they are all real, α is said to be a totally real algebraic number. If the minimal polynomial of α is monic, α is said to be an algebraic integer.

1.1 Kronecker's theorem

Note that if $Q(\alpha) = 0$ for *some* (not necessarily minimal) monic polynomial $Q(x)$ with integral coefficients, then α is an algebraic integer; this is an immediate consequence of the Gauss lemma. Examples of algebraic integers are

given by roots of unity $\zeta_n^k = e^{2\pi ik/n}$ for $k = 1, \dots, n$; they are zeros of the monic polynomial $x^n - 1$.

Exercise 1.1 Show that, for two algebraic integers α and β, their sum $\alpha+\beta$ and product $\alpha\beta$ are algebraic integers as well.

Proposition 1.1 (Kronecker's theorem) *If $\alpha_1 = \alpha$ is a non-zero algebraic integer with all its conjugates α_j inside the unit disc, $|\alpha_j| \le 1$ for $j = 1, \dots, d$, then α is a root of unity.*

Proof Let $P(x) = \prod_{j=1}^d (x - \alpha_j) \in \mathbb{Z}[x]$ denote the minimal polynomial of α; it is monic, because α is an algebraic integer. For each $n = 1, 2, \dots$, consider the polynomial $P_n(x) = \prod_{j=1}^d (x - \alpha_j^n)$, whose roots are the nth powers of α_j, $j = 1, \dots, d$. The coefficients of the polynomial $P_n(x^n) = \prod_{j=1}^d \prod_{k=1}^n (x - \alpha_j \zeta_n^k)$ are integral, hence so are the coefficients of $P_n(x)$ itself. On the other hand, the coefficients are absolutely bounded from above in view of $|\alpha_j^n| \le |\alpha_j| \le 1$ for $j = 1, \dots, d$. This means that the set of polynomials $\{P_n(x) : n = 1, 2, \dots\}$ is finite; in other words, $P_n(x) = P_m(x)$ and

$$\{\alpha_1^n, \alpha_2^n, \dots, \alpha_d^n\} = \{\alpha_1^m, \alpha_2^m, \dots, \alpha_d^m\} \tag{1.1}$$

for some positive $n \ne m$. Changing the order of elements in the multi-sets if necessary, we can conclude that for some ℓ, $1 \le \ell \le d$, we have

$$\alpha_1^n = \alpha_2^m,\ \alpha_2^n = \alpha_3^m,\ \dots,\ \alpha_{\ell-1}^n = \alpha_\ell^m,\ \alpha_\ell^n = \alpha_1^m;$$

eliminating $\alpha_2, \dots, \alpha_\ell$ when $\ell > 1$, we arrive at $\alpha_1^{n^\ell} = \alpha_1^{m^\ell}$, so that $\alpha = \alpha_1$ is an $(m^\ell - n^\ell)$th root of unity. □

Exercise 1.2 If $\alpha^n = \beta^m$ for an algebraic number $\alpha \ne 0$ and its conjugate β, where integers n, m are such that $|n| \ne |m|$, then α is a root of unity.

Hint If $\alpha_1, \dots, \alpha_d$ are algebraic conjugates of α (and β), the Galois group of $\mathbb{Q}(\alpha_1, \dots, \alpha_d)$ over $\mathbb{Q}$ extends the single equality $\alpha^n = \beta^m$ to the equality of multi-sets (1.1). □

A polynomial is said to be cyclotomic if all its roots are roots of unity. The minimal polynomial of an nth root of unity $\zeta_n^k = e^{2\pi ik/n}$, where $\gcd(k, n) = 1$, is given by the nth cyclotomic polynomial

$$\Phi_n(x) = \prod_{\substack{k=1 \\ \gcd(k,n)=1}}^{n} (x - \zeta_n^k)$$

of degree $\varphi(n)$, Euler's totient function.

Exercise 1.3 Show that

$$x^n - 1 = \prod_{d|n} \Phi_d(x),$$

where the product is taken over all positive divisors d of n. In particular, the polynomial

$$\frac{x^n - 1}{x - 1} = x^{n-1} + x^{n-2} + \cdots + x + 1, \quad n = 2, 3, \ldots$$

is irreducible if and only if n is prime.

For a polynomial $P(x)$ of degree d, write $P^*(x) = x^d P(1/x)$. If $P(x) = P^*(x)$, we say that $P(x)$ is reciprocal.

Exercise 1.4 Show that a cyclotomic polynomial $P(x)$ satisfies $P^*(x) = \pm P(x)$. If, in addition, $P(1) \neq 0$, then $P(x)$ is reciprocal.

1.2 Factorisation of cyclotomic expressions

It is a classical fact (cf. also Exercise 1.3 with $x = 2$) that all primes of the form $2^n - 1$, $n = 2, 3, \ldots$, correspond to prime $n = p$. On the other hand, the primality of n does not guarantee $2^n - 1$ to be prime, as the following example demonstrates: $2^{11} - 1 = 23 \cdot 89$. *Primes* of the form $M_p = 2^p - 1$ for p prime are known as Mersenne primes; it remains open whether there are infinitely or finitely many such primes. There are 51 Mersenne primes now known (as of February 2019), with the largest one $M_{82,589,933}$ found in December 2018 [138]; it also happens to be the largest prime known. The latter fact is not a coincidence: primes originating from the values of cyclotomic polynomials possess special primality testing.

Exercise 1.5 If p is an odd prime, then any prime q that divides $2^p - 1$ must be congruent to ± 1 mod 8.

Exercise 1.6 (Lucas–Lehmer test) Define the sequence $s_0 = 4$, s_1, s_2, ... recursively by setting $s_n = s_{n-1}^2 - 2$ for $n \geq 1$. For p an odd prime, show that the number $M_p = 2^p - 1$ is prime if and only if $s_{p-2} \equiv 0 \bmod M_p$.

Based on the earlier work of Pierce [152], in 1933 Derrick Henry Lehmer [122] developed primality testing of numbers of the form

$$\Delta_n(P) = \prod_{j=1}^{d} (\alpha_j^n - 1),$$

where $P(x) = \prod_{j=1}^{d} (x - \alpha_j)$ is a monic polynomial with integral coefficients.

In particular, he demonstrated that

$$\Delta_{113}(x^3 - x - 1) = 63{,}088{,}004{,}325{,}217$$

and

$$\Delta_{127}(x^3 - x - 1) = 3{,}233{,}514{,}251{,}032{,}733$$

are primes. Note that in the proof of Proposition 1.1, $\Delta_n(P)$ were cast as $\pm P_n(1)$.

Theorem 1.2 ([122]) *For $P \in \mathbb{Z}[x]$ monic, $P(0)P(1) \neq 0$, the growth of integers $\Delta_n(P)$ as $n \to \infty$ is given by*

$$\mathrm{M}(P) = \limsup_{n\to\infty} |\Delta_n(P)|^{1/n} = \prod_{j=1}^{d} \max\{1, |\alpha_j|\},$$

where $\alpha_1, \dots, \alpha_d$ are the zeros of $P(x)$.

Furthermore, if the sequence $\Delta_n(P)$, $n = 1, 2, \dots$ is not periodic, then the absolute value of $\Delta_n(P)$ unboundedly increases with n.

Proof The first part follows from

$$\limsup_{n\to\infty} |\alpha^n - 1|^{1/n} = \max\{1, |\alpha|\}$$

for $\alpha \neq 1$.

If the sequence $|\Delta_n(P)|$ (of integers), $n = 1, 2, \dots$, is bounded, then the limit superior is 1, hence $|\alpha_j| \leq 1$ from the first part of the theorem. By Proposition 1.1 the polynomial $P(x)$ is cyclotomic, hence $|\Delta_n(P)| = |P_n(1)|$ is periodic. □

Exercise 1.7 Show that if $P(x)$ is a reciprocal polynomial, then $|\Delta_n(P)|/|\Delta_1(P)|$ is the square of an integer for all odd n, and $|\Delta_n(P)/\Delta_2(P)|$ is the square of an integer for all even n.

Hint (Lalín) Use $\alpha^n - 1 = \alpha^{n/2}(\alpha^{n/2} - \alpha^{-n/2})$ for n even and

$$\frac{\alpha^n - 1}{\alpha - 1} = \alpha^{(n-1)/2} \sum_{j=-(n-1)/2}^{(n-1)/2} \alpha^j$$

for n odd, and the fact that if α is a zero of a reciprocal polynomial $P(x)$ then so is $1/\alpha$. □

Lehmer further suggests [122] some heuristics that in order to obtain large primes from the factorisation of $\Delta_n(P)$, it is advantageous to have the increase of the sequence very slow, in other words, to have $\mathrm{M}(P)$ in Theorem 1.2 as small as possible.

Lehmer's problem If ε is a positive quantity, find a monic polynomial $P(x)$ with integer coefficients such that the absolute value of the product of those roots of P which lie outside the unit circle, lies between 1 and $1+\varepsilon$.

Lehmer gives an example of a small such value; namely, he records [122] (as early as in 1933)

$$\mathrm{M}(x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1) = 1.17628081\ldots. \tag{1.2}$$

This still stands as the smallest value of $\mathrm{M}(P)>1$, in spite of extensive computation done since 1933 by many mathematicians. At the same time, the problem itself remains open.

Here we give a simple argument on the bound of $\mathrm{M}(P)$ from below in the case that the degree of $P(x)$ is bounded.

Lemma 1.3 *For every $d\in\mathbb{Z}_{>0}$ there exists $\mu_d>1$ such that if $P(x)\in\mathbb{Z}[x]$ is a monic, irreducible and non-cyclotomic polynomial of degree d, then $\mathrm{M}(P)\ge\mu_d$.*

Proof Since $\mathrm{M}(x^d-2)=2$, it is sufficient to show that there are *finitely* many monic polynomials $P(x)\in\mathbb{Z}[x]$ of degree d for which $\mathrm{M}(P)<2$. Then μ_d is the minimum of $\mathrm{M}(P)$ over the finite set.

For a polynomial $P(x)=\prod_{j=1}^d(x-\alpha_j)$, the bound $\mathrm{M}(P)=\prod_{1\le j\le d}\max\{1,|\alpha_j|\}<2$ implies $|\alpha_j|<2$ for $j=1,\ldots,d$. As the *integral* coefficients of $P(x)$ are elementary symmetric polynomials of $\alpha_1,\ldots,\alpha_d$, the estimates impose certain bounds on the absolute value of the coefficients of $P(x)$. This implies that the set of such polynomials is indeed finite. □

We conclude this section with a problem related to another classical sequence of Fermat primes — primes of the form $2^{2^n}+1$, $n=0,1,2,\ldots$.

Exercise 1.8 ([155, Division 8, Chapter 2, Problem 94]) Show that all terms of the sequence

$$2^1+1,\ 2^2+1,\ 2^4+1,\ 2^8+1,\ \ldots,\ 2^{2^n}+1,\ \ldots$$

are pairwise coprime.

In fact, this exercise shows that the number of primes that do not exceed a given $x\ge 3$, is at least $c\log\log x$ for an absolute constant $c>0$.

1.3 Jensen's formula

Proposition 1.4 (Jensen's formula [107]) *The following evaluation takes place*:

$$\int_0^1 \log|e^{2\pi it}-\alpha|\,\mathrm{d}t = \log^+|\alpha|,$$

where $\log^+|\alpha| = \max\{0,\log|\alpha|\} = \log\max\{1,|\alpha|\}$.

The result is trivially true for $\alpha = 0$ (when the logarithm is supposed to be $-\infty$), so that we assume $|\alpha| > 0$ below.

Complex-analytic proof Changing to the complex variable $x = e^{2\pi it}$,

$$\int_0^1 \log|e^{2\pi it}-\alpha|\,\mathrm{d}t = \frac{1}{2\pi i}\oint_{|x|=1}\log|x-\alpha|\,\frac{\mathrm{d}x}{x} = \mathrm{Re}\Big(\frac{1}{2\pi i}\oint_{|x|=1}\log(x-\alpha)\,\frac{\mathrm{d}x}{x}\Big).$$

If $|\alpha| > 1$, then $f(x) = \log(x-\alpha)$ is analytic inside the unit disc $|x| < 1$ and on its boundary, hence Cauchy's theorem implies that the latter integral evaluates to $f(0) = \mathrm{Re}\log\alpha = \log|\alpha|$.

If $|\alpha| = 1$, so that the integral is improper, we replace the integration path around $x = \alpha$ with an arc C of radius ε, where $0 < \varepsilon < \frac{1}{2}$, centred at this point lying entirely inside the disc $|x| \le 1$, and use Cauchy's theorem for the newer contour as well as the estimate

$$\Big|\frac{1}{2\pi i}\int_C \log(x-\alpha)\,\frac{\mathrm{d}x}{x}\Big| \le \varepsilon \max_{x:|x-\alpha|=\varepsilon}\frac{|\log(x-\alpha)|}{|x|} \le 2\varepsilon|\log\varepsilon| \to 0 \quad \text{as } \varepsilon\to 0.$$

Finally, if $|\alpha| < 1$, then $\log|x-\alpha| = \log|1-\alpha/x|$ on the contour of integration $|x| = 1$, so that

$$\begin{aligned}\int_0^1 \log|e^{2\pi it}-\alpha|\,\mathrm{d}t &= \mathrm{Re}\Big(\frac{1}{2\pi i}\oint_{|x|=1}\log\Big(1-\frac{\alpha}{x}\Big)\frac{\mathrm{d}x}{x}\Big)\\ &= \mathrm{Re}\Big(\frac{1}{2\pi i}\oint_{|y|=1}\log(1-\alpha y)\,\frac{\mathrm{d}y}{y}\Big).\end{aligned}$$

It remains to observe that the integrand $y^{-1}\log(1-\alpha y)$ has a removable singularity at $y = 0$ and no other within the disc $|y| \le 1$, implying that the integral indeed evaluates to 0. □

Real-analytic proof Write $re^{2\pi is}$ for α and shift t by s to reduce the integral under consideration to

$$\int_0^1 \log|e^{2\pi it}-\alpha|\,\mathrm{d}t = \int_0^1 \log|e^{2\pi it}-r|\,\mathrm{d}t,$$

where $r = |\alpha| > 0$. The resulting integral is seen to be the limit of the integral sums,

$$\int_0^1 \log|e^{2\pi it} - r|\,\mathrm{d}t = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \log|e^{2\pi ik/n} - r|$$
$$= \lim_{n\to\infty} \frac{1}{n} \log\left|\prod_{k=1}^{n} (e^{2\pi ik/n} - r)\right| = \lim_{n\to\infty} \frac{\log|1 - r^n|}{n},$$

and the desired formula follows. □

Exercise 1.9 Show the 'real-looking' version of Jensen's formula:

$$\frac{1}{2} \int_0^1 \log(1 - 2r\cos 2\pi t + r^2)\,\mathrm{d}t = \log^+ r.$$

Hint Notice that $1 - 2r\cos 2\pi t + r^2 = |e^{2\pi it} - r|^2$. □

Let us follow Mahler's steps [129] and replace the linear polynomial $x - \alpha$ with a general polynomial $P(x) = a_d \prod_{j=1}^{d}(x - \alpha_j)$:

$$\int_0^1 \log|P(e^{2\pi it})|\,\mathrm{d}t = \frac{1}{2\pi i} \oint_{|x|=1} \log|a_d| \frac{\mathrm{d}x}{x} + \sum_{j=1}^{d} \frac{1}{2\pi i} \oint_{|x|=1} \log|x - \alpha_j| \frac{\mathrm{d}x}{x}$$
$$= \log|a_d| + \sum_{j=1}^{d} \log\max\{1, |\alpha_j|\}.$$

Proposition 1.5 ([129]) *For a polynomial* $P(x) = a_d \prod_{j=1}^{d}(x - \alpha_j)$,

$$\int_0^1 \log|P(e^{2\pi it})|\,\mathrm{d}t = \frac{1}{2\pi i} \oint_{|x|=1} \log|P(x)| \frac{\mathrm{d}x}{x} = \log|a_d| + \sum_{j=1}^{d} \log^+|\alpha_j|.$$

Comparing this result to the characteristic witnessed in Theorem 1.2, we define the Mahler measure of a polynomial $P(x) = a_d \prod_{j=1}^{d}(x - \alpha_j) \in \mathbb{C}[x]$ as

$$\mathrm{M}(P) = |a_d| \prod_{j=1}^{d} \max\{1, |\alpha_j|\}. \tag{1.3}$$

Then

$$\mathrm{M}(P) = \exp\left(\int_0^1 \log|P(e^{2\pi it})|\,\mathrm{d}t\right) = \exp\left(\frac{1}{2\pi i} \oint_{|x|=1} \log|P(x)| \frac{\mathrm{d}x}{x}\right) \tag{1.4}$$

is the Mahler measure of the polynomial $P(x)$, while the integrals in Proposition 1.5 represent the logarithmic Mahler measure

$$\mathrm{m}(P) = \log \mathrm{M}(P) = \int_0^1 \log|P(e^{2\pi it})|\,\mathrm{d}t = \frac{1}{2\pi i} \oint_{|x|=1} \log|P(x)| \frac{\mathrm{d}x}{x}.$$

The adjective 'logarithmic' is often dropped when use of $\mathrm{m}(\cdot)$ rather than of $\mathrm{M}(\cdot)$ is clear from the context.

Note that the definition implies $\mathrm{M}(PQ) = \mathrm{M}(P) \cdot \mathrm{M}(Q)$ for polynomials $P(x)$ and $Q(x)$ — in other words, the Mahler measure is multiplicative. We can even extend it unambiguously from polynomials to rational functions by setting $\mathrm{M}(P/Q) = \mathrm{M}(P)/\mathrm{M}(Q)$.

For polynomials $P(x)$ with *integer* coefficients, clearly $\mathrm{M}(P) \geq 1$ with $\mathrm{M}(P) = 1$ only if P is monic ($a_d = 1$) and has all its zeros inside the unit circle (hence is a product of a monomial x^n and a cyclotomic polynomial, by Kronecker's theorem).

Exercise 1.10 Show that if $P(x) \in \mathbb{Z}[x]$ then its Mahler measure $\mathrm{M}(P)$ is an algebraic *integer*.

Lemma 1.6 *For a polynomial* $P(x) = a_d x^d + \cdots + a_1 x + a_0 \in \mathbb{C}[x]$,

$$|a_n| \leq \binom{d}{n} \mathrm{M}(P), \quad \textit{where} \ \ n = 0, 1, \ldots, d.$$

Proof According to Viète's theorem, the coefficient a_{d-n} of $P(x)$ is, up to sign, the sum of terms of the form $a_d \alpha_{j_1} \cdots \alpha_{j_n}$, where $\alpha_1, \ldots, \alpha_d$ are the zeros (counted with multiplicity) of $P(x)$. It readily follows from (1.3) that each such term is bounded from above by $\mathrm{M}(P)$, and the result then follows from the fact that the number of terms is exactly $\binom{d}{d-n} = \binom{d}{n}$. □

Lemma 1.7 *For a polynomial* $P(x) = a_d x^d + \cdots + a_1 x + a_0 \in \mathbb{C}[x]$,

$$\mathrm{M}(P) \leq L(P) = |a_0| + |a_1| + \cdots + |a_d|, \quad \textit{the length of } P(x).$$

Proof The estimate follows from Proposition 1.5 and the trivial estimate $|P(x)| \leq L(P)$ for x on the unit circle, $|x| = 1$. □

Exercise 1.11 (Mignotte [139]) Define the norm of polynomial $P(x) = a_d x^d + \cdots + a_1 x + a_0 \in \mathbb{C}[x]$ by

$$\|P(x)\| = (|a_d|^2 + \cdots + |a_1|^2 + |a_0|^2)^{1/2} = \left(\int_0^1 |P(e^{2\pi i t})|^2 \mathrm{d}t \right)^{1/2}.$$

(a) Show that $\mathrm{M}(P) \leq \|P(x)\|$.

(b) If $P(x) = a_d x^d + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ divides a polynomial $Q(x)$ with *integer* coefficients then

$$|a_n| \leq \binom{d}{n} \|Q(x)\| \quad \text{for} \ \ n = 0, 1, \ldots, d.$$

The latter bounds, known as the Landau–Mignotte bounds, allow one to search for potential factors of a given polynomial $Q(x) \in \mathbb{Z}[x]$.

Proposition 1.8 (Gelfond's inequality) *For a product $P(x) = P_1(x)\cdots P_m(x)$ of polynomials, we have*

$$L(P) \le L(P_1)\cdots L(P_m) \le 2^{\deg P} L(P).$$

Proof The lower bound for $L(P_1)\cdots L(P_m)$ is nearly trivial: it does not involve the Mahler measure at all.

In the other direction, it follows from Lemma 1.6 that, for a polynomial $P(x) = a_d x^d + \cdots + a_1 x + a_0$,

$$L(P) = \sum_{n=0}^{d} |a_n| \le \sum_{n=0}^{d} \binom{d}{n} \mathrm{M}(P) = 2^d \mathrm{M}(P).$$

Writing this inequality for each factor $P_j(x)$ and using $\sum_{j=1}^{m} \deg P_j = \deg P$, $\mathrm{M}(P_1\cdots P_m) = \mathrm{M}(P)$ and the bound from Lemma 1.7, we arrive at the required claim. □

Finally, we define the Mahler measure and logarithmic Mahler measure of an algebraic number α as the corresponding quantities for the minimal polynomial $P(x) \in \mathbb{Z}[x]$ of α. In particular, $\mathrm{M}(\alpha) \ge 1$ for an algebraic number α with the equality happening only when $\alpha = 0$ or $\alpha^n = 1$ for some $n \in \mathbb{Z}_{>0}$.

Exercise 1.12 The house $\overline{|\alpha|}$ of an algebraic integer α of degree d is defined as the maximum modulus of its conjugates (including α itself).

(a) Show that $\mathrm{M}(\alpha)^{1/d} \le \overline{|\alpha|} \le \mathrm{M}(\alpha)$.

(b) If $\mathrm{M}(\alpha) < 2$, prove a refined estimate $\mathrm{M}(\alpha) \le (\max\{\overline{|\alpha|}, \overline{|1/\alpha|}\})^{d/2}$.

Hint (b) Observe that in this case $\mathrm{M}(\alpha) = \mathrm{M}(\alpha^{-1})$. □

1.4 Families of Mahler measures

Before going on with partial resolutions of Lehmer's problem, we treat the 'baby' families

$$\lambda^-(k) = \mathrm{m}\Big(x - \frac{1}{x} + k\Big) = \mathrm{m}(x^2 + kx - 1) = \log\frac{k + \sqrt{k^2+4}}{2}$$

and

$$\lambda^+(k) = \mathrm{m}\Big(x + \frac{1}{x} + k\Big) = \mathrm{m}(x^2 + kx + 1) = \log\frac{k + \sqrt{k^2-4}}{2},$$

where k is a positive integer (and $k \geq 3$ in the latter case), and their classical links to the arithmetic of real quadratic fields. Observe that the numbers

$$\varepsilon_k^- = \frac{k + \sqrt{k^2+4}}{2} \quad \text{and} \quad \varepsilon_k^+ = \frac{k + \sqrt{k^2-4}}{2}$$

are units in the corresponding real quadratic fields $\mathbb{Q}(\sqrt{k^2+4})$ and $\mathbb{Q}(\sqrt{k^2-4})$, because their norms are equal to ± 1 (a consequence of the underlying quadratic polynomials). In fact, the purely periodic continued fraction

$$\varepsilon_k^- = k + \cfrac{1}{k + \cfrac{1}{k + \ddots}}$$

for the first unit implies that $\varepsilon_k^- > 1$ is the fundamental unit of $\mathbb{Q}(\sqrt{k^2+4})$ for odd $k > 0$ (see, for example, [33, Chapter 4]).

The structure of a real quadratic field $K = \mathbb{Q}(\sqrt{D})$ is particularly simple: the regulator is given by the logarithm of its fundamental unit $\varepsilon_D > 1$, and the latter generates (up to sign) the multiplicative group of units of K. Applied to our situations, this leads to the statements

$$\lambda^-(k) = r_k^- \cdot \operatorname{Reg} \mathbb{Q}(\sqrt{k^2+4}) \quad \text{and} \quad \lambda^+(k) = r_k^+ \cdot \operatorname{Reg} \mathbb{Q}(\sqrt{k^2-4})$$

for some positive integers r_k^- and r_k^+; more specifically, $r_k^- = 1$ for k odd.

Now let D be the fundamental discriminant of the field $K = \mathbb{Q}(\sqrt{k^2+4})$ or $\mathbb{Q}(\sqrt{k^2-4})$, so that $D = k^2+4$ or k^2-4, except for the situation when $k \equiv 2 \bmod 4$ in the second case and D is $(k^2-4)/4^\ell$ for ℓ such that $(k^2-4)/4^{\ell+1}$ is not divisible by 4. Denote by $\chi(n) = \chi_D(n) = (\frac{D}{n})$ the corresponding Dirichlet character, where $(\frac{D}{\cdot})$ is the Kronecker symbol. Then the Dirichlet class number formula for the field K asserts that

$$\sqrt{D}\, L(\chi, 1) = h(D) \log \varepsilon_D,$$

where $h(D)$ is the number of equivalence classes of quadratic forms with discriminant D and

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

stands for the Dirichlet L-function. Restricted to our situations, this gives

$$\lambda^-(k) = \tilde{r}_k^- \cdot \sqrt{k^2+4}\, L(\chi, 1) \quad \text{and} \quad \lambda^+(k) = \tilde{r}_k^+ \cdot \sqrt{k^2-4}\, L(\chi, 1)$$

for some positive rationals $\tilde{r}_k^-$ and $\tilde{r}_k^+$.

The fact that the quantities $\sqrt{D}\, L(\chi_D, 1)$ and $\operatorname{Reg} K$, for a real quadratic field $K = \mathbb{Q}(\sqrt{D})$, are rationally proportional to each other is classical and follows