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Constrained Willmore Surfaces

Symmetries of a Möbius Invariant Integrable System

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To my parents

Contents

Preface

Among the classes of Riemannian submanifolds, there is that of Willmore surfaces, named after Thomas Willmore (1919–2005), although the topic had already made its appearance early in the nineteenth century, through the works of Marie-Sophie Germain [34, 35] and Siméon Poisson [60], namely their pioneering studies on elasticity and the vibrating properties of thin plates, and again in the 1920s, through the works of Wilhelm Blaschke [5] and Gerhard Thomsen [68], whose findings were forgotten and only brought to light in the 1960s, when the work of Wilmore [72] inspired a renewed interest in the subject, in part due to the celebrated Willmore conjecture, now affirmed by Marques–Neves [51].

From the early 1960s, Willmore devoted particular attention to the quest for the optimal immersion of a given closed surface in Euclidean 3-space, regarding the minimization of some natural energy, motivated by questions on the elasticity of biological membranes and the energetic cost associated with membrane-bending deformation. The Willmore energy of a surface in $\mathbb{R}^3$ is given by its total squared mean curvature. Willmore surfaces are the critical points of the Willmore functional, characterized by the harmonicity of the mean curvature sphere congruence, as established in a key result due to Blaschke [5]. A larger class of surfaces arises when one imposes the weaker requirement that a surface is a critical point of the Willmore functional only with respect to infinitesimally conformal variations: These are the constrained Willmore surfaces.

This work is dedicated to the Möbius invariant class of constrained Willmore surfaces in space-forms and its symmetries. Characterized by the *perturbed harmonicity* of the mean curvature sphere congruence, a generalization of the well-developed integrable systems theory of harmonic maps emerges. The starting point is a zero-curvature characterization, due to

Burstall–Calderbank [12], which we derive from the underlying variational problem. Constrained Willmore surfaces come equipped with a family of flat metric connections. We then define a *spectral deformation* by the action of a loop of flat metric connections, *Bäcklund transformations* by the application of a version of the Terng–Uhlenbeck [67] dressing action by simple factors, and, in 4-space, *Darboux transformations*, based on the solution of a Riccati equation, generalizing the transformation of Willmore surfaces presented in the quaternionic setting by Burstall–Ferus–Leschke–Pedit–Pinkall [16]. We establish a permutability between constrained Willmore spectral deformation and Bäcklund transformation and prove that nontrivial Darboux transformation of constrained Willmore surfaces in 4-space can be obtained as a particular case of Bäcklund transformation. All these transformations corresponding to the zero *Lagrange multiplier* prove to preserve the class of Willmore surfaces.

We dedicate Chapter 8, Section 8.2 to the very special class of constant mean curvature (CMC) surfaces. A classical result by Thomsen [68] characterizes isothermic Willmore surfaces in 3-space as minimal surfaces in some 3-dimensional space-form. Constant mean curvature surfaces in 3-dimensional space-forms are examples of constrained Willmore surfaces, characterized by the existence of some *conserved quantity*. Both constrained Willmore spectral deformation and Bäcklund transformation prove to preserve the existence of such a conserved quantity, defining, in particular, transformations within the class of CMC surfaces in 3-dimensional space-forms, with, furthermore, preservation of both the space-form and the mean curvature, in the latter case. The class of CMC surfaces in 3-dimensional space-forms lies, in this way, at the intersection of several integrable geometries, with classical transformations of its own, as well as transformations as a class of constrained Willmore surfaces, together with transformations as a subclass of the class of isothermic surfaces. Constrained Willmore transformation proves to be unifying to this rich transformation theory, as we shall conclude.

From Bäcklund to Darboux, a comprehensive journey through the transformation theory of constrained Willmore surfaces, with applications to CMC surfaces, this book aims to offer a detailed, self-contained account of the topics explored and the computations involved. It intends to remain as a reader-friendly contribution to the field of integrable systems in Riemannian geometry, serving as both a comprehensive introduction to newcomers and a reference work for researchers.

The present monograph is based on the Ph.D. thesis with the same title submitted to the University of Bath, Department of Mathematical Sciences, in September 2008, and defended viva voce in February 2009. The contents of the thesis have been preserved, with the exception of Section 8.2, where the original study of constrained Willmore spectral deformation and Bäcklund transformation of CMC surfaces has been extended to a general Lagrange multiplier. The current list of references reflects, naturally, the publications that have since taken place.

I would like to take this opportunity to express my deepest gratitude to Professor Francis Burstall, my Ph.D. supervisor, for his constant support and attention, for sharing with me some of his outstanding knowledge of mathematics and promising ideas, and for doing so with the articulation and the enthusiasm that are so distinctive of him. Fran is a great mathematician and an exceptional human being, and it was an absolute privilege and a pleasure to have had the opportunity to learn and work with him throughout my Ph.D. years. I am, and will forever be, deeply grateful to Fran.

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