CAMBRIDGE
Cambridge University Press
978-1-108-79442-8 — Constrained Willmore Surfaces
Áurea Casinhas Quintino
Excerpt
More Information

Introduction

A central theme in Mathematics is that of the search for the optimal representative within a certain class of objects, often driven by the minimization of some energy, reflecting what occurs in many physical processes. From the early 1960s, Thomas Willmore devoted particular attention to the quest for the optimal immersion of a given closed surface in Euclidean 3-space, regarding the minimization of some natural energy, motivated by questions on the elasticity of biological membranes and the energetic cost associated with membrane bending deformations.

We can characterize how much a membrane is bent at a particular point on the membrane by means of the curvature of the osculating circles of the planar curves obtained as perpendicular cross sections through the point. The curvature of these circles consists of the inverse of their radii, with a positive or negative sign depending on whether the membrane curves upwards or downwards, respectively. The minimal and maximal values of the radii of the osculating circles associated with a particular point on the membrane define the principal curvatures, k_1 and k_2, and, from these, the mean curvature, $H = (k_1 + k_2)/2$, and the Gaussian curvature, $K = k_1 k_2$, at the point.

In modern literature on the elasticity of membranes, a weighted sum of the total mean curvature, the total squared mean curvature and the total Gaussian curvature,

$$a \int H + b \int H^2 + c \int K,$$

is considered to be the elastic bending energy of a membrane. By physical considerations, the total mean curvature is neglected. On the other hand, from the perspective of critical points of energy, in deformations preserving the topological type, the total Gaussian curvature can be ignored, according to Gauss–Bonnet theorem. What is left is what Willmore considered to be the elastic bending energy of a compact, oriented Riemannian surface M, without boundary, isometrically immersed in $\mathbb{R}^3$,

$$\mathcal{W} = \int_M H^2 dA,$$

nowadays known as the *Willmore energy*.

The Willmore energy had already made its appearance early in the nineteenth century, through the works of Marie-Sophie Germain [34, 35] and Siméon Poisson [60] and their pioneering studies on elasticity and vibrating properties of thin plates, with the claim that the elastic force of a thin plate is proportional to its mean curvature. Since then, the mean curvature has remained a key concept in the theory of elasticity. The Willmore energy appeared again in the 1920s, in the works of Wilhelm Blaschke [5] and Gerhard Thomsen [68], but their findings were forgotten and only brought to light after the increased interest on the subject motivated by the work of Willmore.

A very interesting fact about the Willmore energy is that it is scale-invariant: if one dilates the surface by any factor, the Willmore energy remains the same. Think of a round sphere in $\mathbb{R}^3$ as an example: if one increases the radius, the surface becomes flatter and its squared mean curvature decreases, but, at the same time, the surface area gets larger, which increases the value of the total squared mean curvature over the surface. One can show that these two phenomena counterbalance each other on any surface. In fact, the Willmore energy has the remarkable property of being invariant under any conformal transformation of $\mathbb{R}^3$, as established in a paper by James White [71] and, actually, already known to Blaschke [5] and Thomsen [68].

From the perspective of critical points of energy, the Willmore functional can be extended to compact, oriented Riemannian surfaces isometrically immersed in a general Riemannian manifold $\hat{M}$ with constant sectional curvature, or *space-form*, by means of

$$\mathcal{W} = \int_M |\Pi_0|^2 dA,$$

the total squared norm of the trace-free part Π_0 of the second fundamental form: by the Gauss equation, relating the curvature tensors of M and $\hat{M}$, we have

$$|\Pi_0|^2 = 2(|\mathcal{H}|^2 - K + \hat{K}),$$

for $\mathcal{H}$ the mean curvature vector and K and $\hat{K}$ the sectional curvatures of M and $\hat{M}$, respectively, so that, in the particular case of surfaces in $\mathbb{R}^3$,

$$|\Pi_0|^2 = 2(H^2 - K),$$

and, therefore, the two functionals share critical points. *Willmore surfaces* are the critical points of the Willmore functional.

It is well known that the Levi-Civita connection is not a conformal invariant. Although the second fundamental form is not conformally invariant, under a conformal change of the metric, its trace-free part remains invariant, so the respective squared norm and the area element change in inverse ways, leaving the Willmore energy unchanged. There is then no reason for carrying a distinguished metric – instead, we consider a conformal class of metrics.

Our study is one of surfaces in n-dimensional space-forms, with $n \geq 3$, from a conformally invariant point of view.[1] So let S^n be the conformal n-sphere, in which, by stereographic projection, we find, in particular, the Euclidean n-space, as well as two copies of hyperbolic n-space. Our surfaces are immersions

$$\Lambda : M \to S^n$$

of a compact, oriented surface M, which we provide with the conformal structure $\mathcal{C}_\Lambda$ induced by Λ and with the canonical complex structure (that is, 90° rotation in the positive direction in tangent spaces, a notion that is, obviously, invariant under conformal changes of the metric). We find a convenient setting in Jean-Gaston Darboux light cone model of the conformal n-sphere [26]. So consider the Lorentzian space $\mathbb{R}^{n+1,1}$ and its light cone $\mathcal{L}$, and fix a unit timelike vector t_0. We identify $v \in S^n \subseteq \mathbb{R}^{n+1}$ with the light-line through $v + t_0$, identifying, in this way, S^n with the projectivized light cone,

$$S^n \cong \mathbb{P}(\mathcal{L}).$$

For us, a surface is, in this way, a null line subbundle $\Lambda = \langle \sigma \rangle$ of the trivial bundle $\underline{\mathbb{R}}^{n+1,1}$ over M, with $\sigma : M \to \mathcal{L}$ a never-zero section of Λ. For further reference, set

$$\Lambda^{1,0} := \Lambda \oplus d\sigma(T^{1,0}M), \qquad \Lambda^{0,1} := \Lambda \oplus d\sigma(T^{0,1}M),$$

independently of the choice of a never-zero $\sigma \in \Gamma(\Lambda)$, and then

$$\Lambda^{(1)} := \Lambda^{1,0} + \Lambda^{0,1}.$$

Along this text, we shall, in general, make no explicit distinction between a bundle and its complexification, and move from real tensors to complex tensors by complex multilinear extension, preserving notation. Our theory is

[1] With the exception of the study of constant mean curvature surfaces, in Section 8.2, which requires carrying a distinguished space-form.

local and, throughout this text, restriction to a suitable nonempty open set shall be underlying. Underlying throughout will be, as well, the identification

$$\wedge^2\mathbb{R}^{n+1,1} \cong o(\mathbb{R}^{n+1,1})$$

of the exterior power $\wedge^2\mathbb{R}^{n+1,1}$ with the orthogonal algebra $o(\mathbb{R}^{n+1,1})$ via

$$u \wedge v(w) := (u, w)v - (v, w)u,$$

for $u, v, w \in \mathbb{R}^{n+1,1}$.

A fundamental construction in conformal geometry of surfaces is the *mean curvature sphere congruence*, or *central sphere congruence*, the bundle of 2-spheres tangent to the surface and sharing with it mean curvature vector at each point. (Although the mean curvature vector is not conformally invariant, under a conformal change of the metric, it changes in the same way for the surface and the osculating 2-sphere.)

In the light cone picture, 2-spheres correspond to $(3, 1)$-planes in $\mathbb{R}^{n+1,1}$ and, in this way, the central sphere congruence defines a map

$$S : M \to \mathrm{Gr}_{(3,1)}(\mathbb{R}^{n+1,1}),$$

into the Grassmannian $\mathcal{G} := \mathrm{Gr}_{(3,1)}(\mathbb{R}^{n+1,1})$ of $(3, 1)$-planes in $\mathbb{R}^{n+1,1}$. We have, therefore, a decomposition $\underline{\mathbb{R}}^{n+1,1} = S \oplus S^\perp$ and then a decomposition of the trivial flat connection d as

$$d = \mathcal{D} + \mathcal{N},$$

for $\mathcal{D}$ the connection given by the sum of the connections induced by d on S and $S^\perp$, respectively, through orthogonal projection.

Given $\mu, \eta \in \Omega^1(S^*T\mathcal{G})$, let $(\mu \wedge \eta)$ be the 2-form defined from the metric on $S^*T\mathcal{G}$:

$$(\mu \wedge \eta)_{(X,Y)} = (\mu_X, \eta_Y) - (\mu_Y, \eta_X),$$

for all $X, Y \in \Gamma(TM)$. The Willmore energy of a surface Λ allows the manifestly conformally invariant formulation given by

$$\mathcal{W}(\Lambda) = \frac{1}{2}\int_M (dS \wedge *dS).$$

It follows the definition presented in [16], in the quaternionic setting, for the particular case of $n = 4$. The intervention of the conformal structure restricts to the Hodge $*$-operator, which is conformally invariant on 1-forms over a surface. The 2-form $(dS \wedge *dS)$ is a conformally invariant way of expressing $|dS|^2 dA$, with respect to any metric in the conformal class $\mathcal{C}_\Lambda$, making clear that the Willmore energy of Λ coincides with the Dirichlet energy of its central sphere congruence,

$$\mathcal{W}(\Lambda) = E(S),$$

as already known to Blaschke [5].

Harmonic maps are the critical points of the Dirichlet energy functional. Willmore surfaces are closely related to harmonic maps via the central sphere congruence, in a key result established by Blaschke [5] (for $n = 3$) and, independently, Norio Ejiri [32] and Marco Rigoli [64] (for general n):

Theorem [5, 32, 64] *Λ is a Willmore surface if and only if its central sphere congruence S is a harmonic map.*

The well-developed integrable systems theory of harmonic maps into Grassmannians now applies. First of all, it provides a zero-curvature characterization of Willmore surfaces. Indeed, for a map into a Grassmannian, harmonicity amounts to the flatness of a certain family of connections, as established by Karen Uhlenbeck [69], and so does then the Willmore surface condition: Λ is a Willmore surface if and only if

$$d^\lambda := \mathcal{D} + \lambda^{-1}\mathcal{N}^{1,0} + \lambda\mathcal{N}^{0,1}$$

is a flat connection, for all $\lambda \in S^1$.

A larger class of surfaces arises when one imposes the weaker requirement that a surface extremize the Willmore functional only with respect to infinitesimally conformal variations: These are the *constrained Willmore (CW) surfaces*. The introduction of a constraint in the variational problem equips surfaces with *Lagrange multipliers*, as first proven by Francis Burstall, Franz Pedit and Ulrich Pinkall [17] and then given the following manifestly conformally invariant formulation by Burstall and David Calderbank [12]:

Theorem [12, 17] *Λ is a CW surface if and only if there exists a real form $q \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$ such that*[2]

$$d^\lambda_q := \mathcal{D} + \lambda^{-1}\mathcal{N}^{1,0} + \lambda\mathcal{N}^{0,1} + (\lambda^{-2} - 1)q^{1,0} + (\lambda^2 - 1)q^{0,1}$$

is a flat connection, for all $\lambda \in S^1$.

Such a form q is said to be a (Lagrange) multiplier for Λ, and Λ is said to be a q-constrained Willmore surface. At times, it will be convenient to make an explicit reference to the central sphere congruence of Λ, writing $d_S^{\lambda,q}$ for d^λ_q.

[2] In the literature, the associated family of flat connections corresponding to a different choice of orientation of M can also be found.

Willmore surfaces are the CW surfaces admitting the zero multiplier. This is not necessarily the only multiplier:

Theorem *A CW surface Λ admits a unique multiplier if and only if Λ is not an isothermic surface.*

Isothermic surfaces are classically defined by the existence of conformal curvature line coordinates. Conformal curvature line coordinates are preserved under conformal changes of the metric and, therefore, so is the isothermic surface condition, allowing the following manifestly conformally invariant formulation, due to Burstall, Neil Donaldson, Pedit and Pinkall [14]:

Theorem [14] *Λ is an isothermic surface if and only if there exists a non-zero closed real 1-form $\eta \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$. In this case, we say that Λ is a η-isothermic surface.*

As we shall see, if $q_1 \neq q_2$ are multipliers for Λ, then Λ is a $*(q_1 - q_2)$-isothermic surface, and, reciprocally, if Λ is a η-isothermic q-constrained Willmore surface, then the set of multipliers for Λ is the affine space $q + \langle *\eta \rangle_{\mathbb{R}}$.

A classical result by Thomsen [68] characterizes isothermic Willmore surfaces in 3-space as the minimal surfaces in some 3-dimensional space-form. (In contrast to CW surfaces, constant mean curvature (CMC) surfaces are not conformally invariant objects, requiring a distinguished space-form to be considered.)

Theorem [68] *Λ is a minimal surface in some 3-dimensional space-form if and only if Λ is an isothermic Willmore surface in 3-space.*

Minimal surfaces are defined variationally as the stationary configurations for the area functional, among all those spanning a given boundary. These were first considered by Joseph-Louis Lagrange [44], in 1762, who raised the question of the existence of surfaces of least area among all those spanning a given closed curve in Euclidean 3-space as boundary. Earlier, Leonhard Euler [33] had already discussed minimizing properties of the surface now known as the catenoid, although he only considered variations within a certain class of surfaces. The problem raised by Lagrange became known as the Plateau's Problem, referring to Joseph Plateau [59], who first experimented with soap films.

A physical model of a minimal surface can be obtained by dipping a wire loop into a soap solution. The resulting soap film is minimal in the sense that it always tries to organize itself so that its surface area is as small as possible while spanning the wire contour. This minimal surface area is, naturally, reached for the flat position, which happens to be a position of vanishing mean curvature. This does not come as a particular feature of this rather simple example of minimal surface. In fact, the Euler–Lagrange equation of the variational problem underlying minimal surfaces turns out to be precisely the zero mean curvature equation, as discovered by Jean Baptiste Meusnier [52]. The flat position of the soap film is also the position in which the membrane is the most relaxed. These surfaces are elastic energy minimals and, in this way, examples of Willmore surfaces.

Unlike flat soap films, soap bubbles exist under a certain surface tension, in an equilibrium where slightly greater pressure inside the bubble is balanced by the area-minimizing forces of the bubble itself. With their spherical shape, soap bubbles are examples of area-minimizing surfaces under the constraint of the volume enclosed. These are surfaces of (nonzero) constant mean curvature. Constant mean curvature surfaces in 3-dimensional space-forms are examples of isothermic CW surfaces, as established by Jörg Richter [63]. The converse, however, is not true, as established by an example due to Burstall, presented in [8], of a CW cylinder that does not have constant mean curvature in any space-form.

The class of CW surfaces in space-forms constitutes a Möbius invariant class of surfaces with strong links to the theory of integrable systems. The zero-curvature characterization of CW surfaces presented earlier allows one to deduce two types of symmetry: a spectral deformation, by the action of a loop of flat metric connections, following the work of Uhlenbeck [69], as well as *Bäcklund transformations*, by the application of a version of the dressing action by simple factors developed by Chuu-Lian Terng and Uhlenbeck [67].

Suppose that Λ is a q-constrained Willmore surface. The two types of transformations that we describe next apply to any choice of the multiplier q (when there is a choice to be made) and depend on it. In the particular case that Λ is a Willmore surface, consider, for the moment, q to be the zero multiplier.

The simplest transformation of Λ into new CW surfaces arises from exploiting a scaling freedom in the spectral parameter, as follows. For each $\lambda \in S^1$, the flatness of the metric connection d_q^λ establishes the existence of an isometry of bundles

$$\phi_\lambda : (\underline{\mathbb{R}}^{n+1,1}, d_q^\lambda) \to (\underline{\mathbb{R}}^{n+1,1}, d),$$

preserving connections, defined on a simply connected component of M and unique up to a Möbius transformation. We define a spectral deformation of Λ by setting, for each $\lambda \in S^1$,

$$\Lambda_\lambda := \phi_\lambda \Lambda.$$

For each $\lambda \in S^1$, set

$$q_\lambda := \phi_\lambda \circ (\lambda^{-2} q^{1,0} + \lambda^2 q^{0,1}) \circ (\phi_\lambda)^{-1}.$$

The central sphere congruence of Λ_λ is $\phi_\lambda S$ and, given $\mu \in S^1$, we have

$$d^{\mu, q_\lambda}_{\phi_\lambda S} = \phi_\lambda \circ d^{\mu\lambda, q}_S \circ (\phi_\lambda)^{-1},$$

establishing the flatness of $d^{\mu, q_\lambda}_{\phi_\lambda S}$ from that of $d^{\mu\lambda, q}_S$ (note that $\mu\lambda \in S^1$). It follows that:

Theorem *Λ_λ is a q_λ-constrained Willmore surface, for all $\lambda \in S^1$.*

In particular, this spectral deformation preserves the zero multiplier.

Corollary *If Λ is a Willmore surface, then so is Λ_λ, for all $\lambda \in S^1$.*

This spectral deformation coincides, up to reparameterization, with the one presented by Burstall–Pedit–Pinkall [17], in terms of the *Hopf differential* and the *Schwarzian derivative*, as we shall verify.

The isothermic surface condition is known [17] to be preserved under CW spectral deformation. In our setting, one can verify that, if Λ is a η-isothermic surface, then Λ_λ is a η_λ-isothermic surface, for

$$\eta_\lambda := \phi_\lambda \circ (\lambda^{-1} \eta^{1,0} + \lambda \eta^{0,1}) \circ (\phi_\lambda)^{-1}.$$

Theorem [17] *If Λ is an isothermic surface, then so is Λ_λ, for all $\lambda \in S^1$.*

Hence:

Corollary *If Λ is a minimal surface in some 3-dimensional space-form, then so is Λ_λ, for each $\lambda \in S^1$ (although not necessarily with preservation of the space-form).*

As we shall see, following the introduction of the notion of *conserved quantity* of a CW surface, this spectral deformation preserves, as well, the class of CMC surfaces in 3-dimensional space-forms, for special choices of the spectral parameter.

Having exploited the equivalence of $d^{\lambda, q}_S$ to the trivial flat connection, as flat metric connections, by means of

$$d^{\lambda, q}_S = (\phi_\lambda)^{-1} \circ d \circ \phi_\lambda,$$

next, we explore gauge equivalences starting from $d^{\lambda, q}_S$, i.e., equivalences given by

$$d^{\lambda, q^*}_{S^*} = r(\lambda) \circ d^{\lambda, q}_S \circ r(\lambda)^{-1},$$

for some q^* and some S^*, with $r(\lambda) \in \Gamma(O(\underline{\mathbb{R}}^{n+1,1}))$, so that the flatness of $d_S^{\lambda,q}$ establishes that of $d_{S^*}^{\lambda,q^*}$. The difficulties involved are of two different orders, namely, the preservation of the algebraic shape of $d_S^{\lambda,q}$, together with ensuring that S^* still is the central sphere congruence of some surface, so that the family of flat connections $d_{S^*}^{\lambda,q^*}$ is the associated family to some CW surface. A version of the Terng–Uhlenbeck [67] dressing action by simple factors proves to offer a simple construction, out of two parameters – a complex number α and a null line bundle L, parallel with respect to $d_S^{\alpha,q}$ – from which we define, respectively, the eigenvalues and the eigenspaces of two different types of linear fractional transformations, whose composition produces a desired gauge transformation r, as follows.

Let ρ denote reflection across S,

$$\rho = \pi_S - \pi_{S^\perp},$$

for π_S and $\pi_{S^\perp}$ the orthogonal projections of $\underline{\mathbb{R}}^{n+1,1}$ onto S and $S^\perp$, respectively. Given $\alpha \in \mathbb{C}$ and L a null line subbundle of $\underline{\mathbb{R}}^{n+1,1}$ such that $\rho L \cap L^\perp = 0$, set

$$p_{\alpha,L}(\lambda) := I \begin{cases} \frac{\alpha-\lambda}{\alpha+\lambda} & \text{on } L \\ 1 & \text{on } (L \oplus \rho L)^\perp \\ \frac{\alpha+\lambda}{\alpha-\lambda} & \text{on } \rho L \end{cases},$$

for $\lambda \in \mathbb{C}\backslash\{\pm\alpha\}$ and $I \in \Gamma(O(\underline{\mathbb{R}}^{n+1,1}))$ the identity map of $\underline{\mathbb{R}}^{n+1,1}$. Let $q_{\alpha,L}$ denote the map obtained from $p_{\alpha,L}$ by considering the additive inverses of the eigenvalues associated to the eigenspaces L and ρL, respectively. Define $p_{\alpha,L}(\infty)$ and $q_{\alpha,L}(\infty)$ by holomorphic extension of

$$p_{\alpha,L}, q_{\alpha,L} : \mathbb{C}\backslash\{\pm\alpha\} \to \Gamma(O(\underline{\mathbb{R}}^{n+1,1})),$$

respectively.

Now, consider $\alpha \in \mathbb{C}\backslash(S^1 \cup \{0\})$ and L a $d_S^{\alpha,q}$-parallel null line subbundle of $\underline{\mathbb{R}}^{n+1,1}$ such that $\rho L \cap L^\perp = 0$ (whose existence can be proved). Set $\alpha^* := \overline{\alpha}^{-1}$, $L' := p_{\alpha,L}(\alpha^*)\overline{L}$ and, for each $\lambda \in \mathbb{P}^1\backslash\{\pm\alpha\}$,

$$r(\lambda) := q_{\alpha^*,L'}(\lambda)\, p_{\alpha,L}(\lambda).$$

Consider the transform

$$S^* := r(1)^{-1} S$$

of S and the transform

$$\Lambda^* := (r(1)^{-1} r(\infty)\Lambda^{1,0}) \cap (r(1)^{-1} r(0)\Lambda^{0,1})$$

of Λ, defining, in particular, a surface with central sphere congruence S^*. Set, furthermore,

$$q^* := r(1)^{-1} \circ (r(0) \circ q^{1,0} \circ r(0)^{-1} + r(\infty) \circ q^{0,1} \circ r(\infty)^{-1}) \circ r(1).$$

Theorem *Λ^* is a q^*-constrained Willmore surface, said to be the* Bäcklund transform of Λ of parameters α, L.[3]

In particular, Bäcklund transformation preserves the zero multiplier.

Corollary *If Λ is a Willmore surface, then so is Λ^*.*

It is not clear that if Λ is an isothermic surface, then so is Λ^*. So far, it is not clear either that Bäcklund transformation preserves the class of minimal surfaces in 3-dimensional space-forms. However, as we shall see later, that proves to be the case. We shall see, furthermore, following the introduction of the notion of *conserved quantity* of a CW surface, that Bäcklund transformation preserves the class of CMC surfaces in 3-dimensional space-forms, for special choices of parameters, with preservation of both the mean curvature and the curvature of space, defining, in particular, a transformation within the class of minimal surfaces in space-forms, with preservation of the space-form.

As established by Burstall–Donaldson–Pedit–Pinkall [14], the isothermic surface condition amounts, just as well, to the flatness of a certain family ∇^t of connections, indexed in $\mathbb{R}$. The theory of ordinary differential equations ensures that one can find ∇^t-parallel sections depending smoothly on the spectral parameter t. The existence of such sections with polynomial dependence on t is of particular geometric significance, as first observed by Burstall–Calderbank [13], and gave rise to the notion of *polynomial conserved quantity*, developed by Burstall and Susana Santos [20] in the isothermic context. We are in this way led to the notion of *conserved quantity* for CW surfaces, presented in Chapter 7.

Let Λ be a q-constrained Willmore surface. A Laurent polynomial

$$p(\lambda) = \lambda^{-1}v + v_0 + \lambda\overline{v},$$

with $v_0 \in \Gamma(S)$ real, $v \in \Gamma(S^\perp)$ and

$$p(1) \neq 0,$$

[3] In [18], an extra factor is introduced in the eigenvalues of $p_{\alpha,L}(\lambda)$, resulting in the normalisation of the family $\lambda \mapsto p_{\alpha,L}(\lambda)$, $p_{\alpha,L}(1) = I$.