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Stability of Tangent Bundles on Smooth Toric Picard-rank-2 Varieties and Surfaces

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To Bill Fulton on the occasion of his 80th birthday.

Abstract. We give a combinatorial criterion for the tangent bundle on a smooth toric variety to be stable with respect to a given polarisation in terms of the corresponding lattice polytope. Furthermore, we show that for a smooth toric surface X and a smooth toric variety of Picard rank 2, there exists an ample line bundle with respect to which the tangent bundle is stable if and only if it is an iterated blow-up of projective space.

1 Introduction

Let X be a smooth toric variety of dimension n over a field of characteristic 0, with tangent bundle $\mathcal{T}_X$. Let $\mathcal{O}(D)$ be an ample line bundle. Recall that the slope of a torsion-free sheaf $\mathcal{E}$ on a normal projective variety X with respect to a nef line bundle $\mathcal{O}(D)$ is defined to be

$$\mu(\mathcal{E}) = \frac{c_1(\mathcal{E}) \cdot D^{n-1}}{\operatorname{rank}(\mathcal{E})},$$

and that $\mathcal{E}$ is *stable* (resp. *semistable*) with respect to $\mathcal{O}(D)$ if for any subsheaf $\mathcal{F}$ of $\mathcal{E}$ of smaller rank, we have $\mu(\mathcal{F}) < \mu(\mathcal{E})$ (resp. $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$). A direct sum of stable sheaves with identical slope is called *polystable*. A situation of particular interest is when X is Fano, $\mathcal{E} = \mathcal{T}_X$ is the tangent bundle, and $D = -K_X$ the anticanonical divisor, in particular, since the existence of a Kähler–Einstein metric on a Fano variety implies that the tangent bundle is polystable with respect to the anticanonical polarisation, see Section 1.1 for more details.

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The main question we discuss in this article is when toric varieties admit a polarisation $\mathcal{O}(D)$ such that the tangent bundle $\mathcal{T}_X$ is stable with respect to $\mathcal{O}(D)$. This question has been studied in [29] and recently also by Biswas, Dey, Genc, and Poddar in [2]. Note that it is well-known that the tangent bundle on projective space is stable with respect to $\mathcal{O}_{\mathbb{P}^n}(1)$.

Theorem 1.1 *Let X be a smooth toric surface or a smooth toric variety of Picard rank 2. Then there exists an ample line bundle $\mathcal{L}$ on X such that T_X is stable with respect to $\mathcal{L}$ if and only if it is an iterated blow-up of projective space.*

For more precise statements, see Theorems 1.3 and 1.4. Theorem 1.4 and a more detailed discussion of the Fano case has been independently obtained by Dasgupta, Dey, and Khan [11]. While for smooth toric varieties of Picard rank 3 it is an open question whether Theorem 1.1 holds, there exists a toric Fano 3-fold of Picard rank 4 whose tangent bundle is stable with respect to the anticanonical polarisation, but that does not admit a morphism to $\mathbb{P}^3$, see Example 5.1.

We deduce the following criterion for the tangent bundle $\mathcal{T}_X$ on a toric variety X to be stable with respect to a given polarisation $\mathcal{O}(D)$ from well-known descriptions of stability conditions in terms of the Klyachko filtrations associated to the tangent bundle (see, for example, [20, 21, 23]). Now fix a fan Σ corresponding to X. Let P_D be the lattice polytope associated to D. For each ray ρ in Σ, let P_D^ρ denote the facet corresponding to ρ, and let v_ρ denote the primitive vector generating ρ.

Proposition 1.2 *The tangent bundle on a smooth projective toric variety X of dimension n is (semi-)stable with respect to an ample line bundle $\mathcal{O}(D)$ on X if and only if for every proper subspace $F \subsetneq N \otimes k$ that is generated by primitive vectors v_ρ generating rays in the fan Σ, the following inequality holds:*

$$\frac{1}{\dim F} \sum_{v_\rho \in F} \mathrm{vol}(P_D^\rho) \overset{(\leq)}{<} \frac{1}{n} \sum_{\rho} \mathrm{vol}(P_D^\rho) = \frac{1}{n} \mathrm{vol}\, \partial P_D. \tag{1}$$

Here, $\mathrm{vol}(P^\rho)$ denotes the lattice volume inside the affine span of P^ρ with respect to the lattice $\mathrm{span}(P^\rho) \cap M$.

We now present our results with more details. Let $\mathrm{Amp}(X) \subset N^1(X)_{\mathbb{R}}$ denote the ample cone of X. It is convenient to define

$$\mathrm{Stab}(\mathcal{T}_X) = \{D \in \mathrm{Amp}(X) \mid \mathcal{T}_X \text{ is stable with respect to } \mathcal{O}(D)\}, \text{ and}$$
$$\mathrm{sStab}(\mathcal{T}_X) = \{D \in \mathrm{Amp}(X) \mid \mathcal{T}_X \text{ is semistable with respect to } \mathcal{O}(D)\}.$$

Using results from [14] one can show that if for a $\mathbb{Q}$-factorial variety $\mathrm{Stab}(\mathcal{T}_X)$ is non-empty, then for any birational morphism $X' \to X$, $\mathrm{Stab}(\mathcal{T}_{X'}) \neq \emptyset$, see 2.7. In particular, since the tangent bundle to $\mathbb{P}^n$ is stable with respect to the anticanonical polarisation, any iterated blow-up of projective space has $\mathrm{Stab}(\mathcal{T}_X) \neq \emptyset$.

Recall that every smooth toric surface is either a successive toric blow-up of $\mathbb{P}^2$ or of a Hirzebruch surface $\mathbb{F}_a$. In Lemma 3.2, we characterise the fans of smooth toric surfaces that are not a blow-up of $\mathbb{P}^2$ and use this to prove the following theorem.

Theorem 1.3 *Let X be a smooth toric surface. Then*

1 $\mathrm{Stab}(\mathcal{T}_X) = \mathrm{Amp}(X)$ *if and only if* $X = \mathbb{P}^2$
2 $\emptyset = \mathrm{Stab}(\mathcal{T}_X) \subsetneq \mathrm{sStab}(\mathcal{T}_X)$ *if and only if* $X \cong \mathbb{P}^1 \times \mathbb{P}^1$.
3 $\emptyset \subsetneq \mathrm{Stab}(\mathcal{T}_X) \subsetneq \mathrm{Amp}(X)$ *if and only if X is an iterated blow-up of* $\mathbb{P}^2$*, but not* $\mathbb{P}^2$ *itself,*
4 $\mathrm{Stab}(\mathcal{T}_X) = \emptyset$ *if and only if X is not an iterated blow-up of* $\mathbb{P}^2$.

In [2, Theorem 6.2], Biswas et al. show that when X is the Hirzebruch surface $\mathbb{F}_a$, $a \geq 2$ implies that $\mathrm{Stab}(\mathcal{T}_X) = \emptyset$ and for $a = 1$ they describe $\mathrm{Stab}(\mathcal{T}_X)$ in [2, Corollary 6.3].

Projectivisations of direct sums of line bundles on projective spaces yield examples of toric Fano varieties under some conditions, but are also interesting in their own right. By [18, Theorem 1] every smooth toric variety of Picard rank 2 is of the form $X = \mathbb{P}_{\mathbb{P}^s}(\mathcal{O} \oplus \bigoplus_{i=1}^r \mathcal{O}(a_i))$, and X is a blow-up of $\mathbb{P}^s$ if and only if $(a_1, \ldots, a_r) = (0, \ldots, 0, 1)$. Note that the polytopes corresponding to ample line bundles on these varieties are special cases of *Cayley polytopes*, see for example [4].

In general, the projectivization $X := \mathbb{P}_Y(\mathcal{E}) = \mathrm{Proj}_Y(\bigoplus_m S^m\mathcal{E})$ of a vector bundle $\mathcal{E}$ on a variety Y admits a relatively ample line bundle $\mathcal{O}_X(1)$ induced by the relative Proj construction and we have $\mathrm{Pic}(X) = \pi^* \mathrm{Pic}(Y) \oplus \mathbb{Z}\mathcal{O}_X(1)$. Here, $\pi : X \to Y$ is the structure morphism of the relative Proj-construction. In the following, we always have $Y = \mathbb{P}^s$ and every element of the Picard group of X can be uniquely written in the form $\mathcal{O}_X(\lambda) \otimes \pi^*\mathcal{O}(\mu)$ with $\lambda, \mu \in \mathbb{Z}$.

Theorem 1.4 *Consider the smooth projective variety*

$$X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^s} \oplus \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^s}(a_i))$$

for $s, r \geq 1$ *with* $0 \leq a_1 \leq \cdots \leq a_r$. *For* $a_r \geq 1$, *we have* $\mathrm{Stab}(\mathcal{T}_X) \neq \emptyset$ *if and only if* $\mathrm{sStab}(\mathcal{T}_X) \neq \emptyset$ *if and only if* $(a_1, \ldots, a_r) = (0, \ldots, 0, 1)$. *In this case,*

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$\mathcal{T}_X$ is (semi-)stable with respect to a polarisation $\mathcal{L} = \mathcal{O}_X(\lambda) \otimes \pi^*\mathcal{O}(\mu)$ if and only if $p(\mu/\lambda) \overset{(\le)}{<} 0$, where $p(x)$ is the following polynomial of degree s:

$$p(x) := -\left(\sum_{q=0}^{s-1} \binom{r+s-1}{q} x^q\right) + \frac{s(r+1)}{r} \binom{r+s-1}{s} x^s.$$

We note that $p(\mu/\lambda) < 0$ if and only if μ/λ is in the interval $(0, \gamma)$, where γ is the only positive root of $p(x)$. For $r = 1$ we have $\gamma = \frac{1}{(2s+1)^{1/s}-1}$, and for $s = 1$ we get $\gamma = \frac{1}{r+1}$.

One has $\emptyset = \mathrm{Stab}(\mathcal{T}_X) \subsetneq \mathrm{sStab}(\mathcal{T}_X)$ if and only if $(a_1, \ldots, a_r) = (0, \ldots, 0)$, i.e. if $X = \mathbb{P}^s \times \mathbb{P}^r$. In this case $\mathcal{T}_X$ is semistable only with respect to pluri-anticanonical polarisations.

This result has been independently proved by [11]. It is extending a result by Biswas et. al. [2, Theorem 8.1], who show that in the Fano case (when $0 < a \le s$), and when $s \ge 2$, the tangent bundle on $X = \mathbb{P}_{\mathbb{P}^s}(\mathcal{O} \oplus \mathcal{O}(a))$ is not stable with respect to the anticanonical polarisation $\mathcal{O}(-K_X) = \mathcal{O}(2) \otimes \pi^*\mathcal{O}(s+1-a)$.

The tangent bundle to a smooth Fano surface is stable with respect to the anticanonical polarisation by [12]. Moreoever, in [31] all smooth Fano threefolds with stable (resp. semistable) tangent bundle are classified. Moreover, for smooth toric Fano varieties of dimension 4 and Picard rank 2, the (semi-)stability of the tangent bundle with respect to the anticanonical polarisation is treated in [2, Section 9], and for smooth toric Fano varieties of dimension 4 and Picard rank 3 in [11].

The above results motivate the following question:

Question 1.5 Are there only finitely many isomorphism classes of smooth projective toric varieties X of given dimension n and Picard number ρ with $\mathrm{Stab}(T_X) \neq \emptyset$?

Corollary 1.6 *Question 1.5 has an affirmative answer for $n \le 2$ or $\rho \le 2$.*

Proof The cases $n = 1$ or $\rho = 1$ are trivial. For $n = 2$ this follows from Theorem 1.3(3). For $\rho = 2$ this follows from Theorem 1.4 (note that $\dim(X) = r + s$). □

1.1 Connections to the existence problem of Kähler-Einstein metrics

When X is a smooth Fano variety over the complex numbers, the existence of a Kähler-Einstein metric on the underlying complex manifold X implies that its tangent bundle is polystable, (in particular, semistable) with respect to the anticanonical polarisation [25], [22, Sec 5.8]. However, the converse does not hold for the blow-up of $\mathbb{P}^2$ in two points [27]. The recent proof of the Yau–Tian–Donaldson conjecture [5, 6, 7, 8] shows that a Fano manifold has a Kähler-Einstein metric if and only if it is K-polystable. For a general toric Fano variety K-stability is equivalent to the fact that for the polytope corresponding to the anticanonical polarisation the barycenter coincides with the origin [24], in the smooth case this was known before due to combining [32] and [26].

Thus we obtain the following combinatorial statement:

Corollary 1.7 *Let P be a smooth reflexive polytope with barycenter in the origin. Then P satisfies the non-strict inequality* (1) *for every proper linear subspace $F \subset N_{\mathbb{Q}}$.*

This statement has been known to combinatorialists in a more general setting that implies the statement for reflexive polytopes with barycenter in the origin (without the smoothness assumption). Conditions of this type are known in convex geometry under the name *subspace concentration conditions*. They play a distinguished role in several problems from convex geometry, see e.g. [17, 16, 15]. The fact that this condition holds for a reflexive polytope whenever the barycenter coincides with the origin is far from being obvious. Moreover, our argument via Kähler-Einstein metrics is valid only in the smooth case (since we have to rely on [25], [22, Sec 5.8]), but the fact turns out to be true for every reflexive polytope. This follows from an even more general result in [15, Thm 1.1], which applies to every polytope with barycentre at the origin. Their proof relies entirely on methods from convex geometry.

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2 Stability conditions for equivariant sheaves

We fix our setting as follows. We consider a polarized toric variety $(X, \mathcal{O}(D))$ corresponding to a lattice polytope P. Let Σ be the normal fan of P and P^ρ the facet of P corresponding to a ray $\rho \in \Sigma$. Denote by $\Sigma(1)$ the set of rays in Σ.

Recall that a coherent sheaf $\mathcal{E}$ is called *reflexive* if $\mathcal{E} \cong \mathcal{E}^{\vee\vee}$, where $\mathcal{E}^\vee = \mathcal{H}om(\mathcal{E}, \mathcal{O}_X)$. In [19] equivariant vector bundles on smooth toric varieties were classified in terms of collections of filtrations of k-vector spaces indexed by the rays of Σ. This classficiation extends to equivariant reflexive sheaves on normal toric varieties, see for example, [21, 30].

More precisely, we fix a k-vector space E and for every ray $\rho \in \Sigma^{(1)}$ we consider a decreasing filtration by subspaces

$$E \supset \cdots \supset E^\rho(i-1) \supset E^\rho(i) \supset E^\rho(i+1) \supset \cdots \supset 0,$$

such that $E^\rho(i)$ differs from E and 0 only for finitely many values of $i \in \mathbb{Z}$. Given such a collection of filtrations for every cone $\sigma \subset \Sigma$ we may consider

$$E_u := \left(\bigcap_{\rho \in \Sigma^{(1)}} E^\rho(-\langle v_\rho, u \rangle) \right) \otimes \chi^u \subset E \otimes k[M].$$

Then $\bigoplus_{u \in M} E_u$ is equipped with the structure of an M-graded $k[U_\sigma]$-module via the natural multiplication with $\chi^u \in k[U_\sigma]$. Then setting $H^0(U_\sigma, \mathcal{E}) = \bigoplus_{u \in M} E_u$ for every $\sigma \in \Sigma$ defines an equivariant reflexive sheaf on X.

The collections of filtrations form an abelian category in a natural way. A morphism between a collection of filtrations $F^\rho(i)$ of a vector space F and another collection $E^\rho(i)$ of filtrations of a vector space E is a linear map $L\colon F \to E$ that is compatible with the filtrations, i.e. $L(F^\rho(i)) \subset E^\rho(i)$ for all $\rho \in \Sigma^{(1)}$ and all $i \in \mathbb{Z}$.

Theorem 2.1 *There is an equivalence of categories between equivariant reflexive sheaves on a toric variety $X = X_\Sigma$ and collections of filtrations of k-vector spaces indexed by the rays of Σ. Here, the rank of the reflexive sheaf equals the dimension of the filtered k-vector space.*

For a collection of filtrations $E^\rho(i)$, we set $e^{[\rho]}(i) = \dim E^\rho(i) - \dim E^\rho(i+1)$. Similarly, for other filtrations we will always use the lower letter version to denote the differences of dimensions between the steps of the filtration. Then we have the following formula.

Lemma 2.2 *Assume that X is smooth. With the notation above we have*

$$\mu(\mathcal{E}) = \frac{1}{\dim E} \sum_{i,\rho} i \cdot e^{[\rho]}(i) \cdot \mathrm{vol}(P^\rho).$$

Proof By [23, Corollary 3.18], we have $c_1(\mathcal{E}) = \sum_\rho \sum_{i\in\mathbb{Z}} i e^{[\rho]}(i) D_\rho$. Now, for a ray $\rho \in \Sigma^{(1)}$ the intersection number $D^{n-1} \cdot D_\rho$ is given by the volume of the corresponding facet P^ρ of P, see e.g. [9]. □

With the notation above we get the following characterisation of stability.

Proposition 2.3 *Let X be a smooth toric variety. A toric vector bundle $\mathcal{E}$ on X corresponding to filtrations $E^\rho(i)$ is (semi-)stable if and only if the following inequality holds for every linear subspace $F \subset E$ and $F^\rho(i) = E^\rho(i) \cap F$.*

$$\frac{1}{\dim F} \sum_{i,\rho} i \cdot f^{[\rho]}(i) \cdot \mathrm{vol}(P^\rho) \overset{(\leq)}{<} \frac{1}{\dim E} \sum_{i,\rho} i \cdot e^{[\rho]}(i) \cdot \mathrm{vol}(P^\rho) \quad (2)$$

Proof By [23, Proposition 4.13] it is sufficient to consider equivariant reflexive subsheaves. It remains to show that it is sufficient to consider those subsheaves, which correspond to filtrations of the form $E^\rho(i) \cap F$. For every subsheaf $\mathcal{F}' \subset \mathcal{E}$ corresponding to filtrations $(F')^\rho(i) \subset E^\rho(i)$ of some subspace $F \subset E$ we may consider the subsheaf $\mathcal{F}$ corresponding to the filtrations $F^\rho(i) := E^\rho(i) \cap F$. Then $\dim F^\rho(i) \geq \dim(F')^\rho(i)$ for all i, ρ. Now, Lemma 2.5 implies that $\mu(\mathcal{F}) \geq \mu(F')$. □

Remark 2.4 A subsheaf $\mathcal{F}$ of a torsion-free sheaf $\mathcal{E}$ is called *saturated* if $\mathcal{E}/\mathcal{F}$ is torsion-free. The saturation of a subsheaf $\mathcal{F} \subset \mathcal{E}$ is the smallest saturated subsheaf of $\mathcal{E}$ containing $\mathcal{F}$. It is not hard to derive from the description of $H^0(U_\sigma, \mathcal{E})$ given above, that $\mathcal{F} \subset \mathcal{E}$ given by $F^\rho(i) \subset E^\rho(i)$ is saturated, if and only if $F^\rho(i) = E^\rho(i) \cap F$. Hence, Lemma 2.5 below can be seen as a combinatorial version of the well-known fact that replacing a subsheaf by its saturation increases the slope.

Lemma 2.5 *Given integer functions $f, g : \mathbb{Z} \to \mathbb{Z}$ with $f \geq g$ such that $\{i \in \mathbb{Z} \mid f(i) \neq g(i)\}$ is finite. Then also*

$$\sum_i i \cdot (f(i) - f(i+1)) \geq \sum_i i \cdot (g(i) - g(i+1))$$

holds.

Proof Note that the assumption implies that $A(f,g) := \sum_i (f(i) - g(i)) \geq 0$ is finite. We fix f and proceed by induction on $A(f,g)$. If $A(f,g) = 0$, $f = g$ and the statement is trivially true. Fix f and assume that the statement holds for all $g \le f$ with $A(f,g) \le A$. Let g' be such that $A(f,g') = A+1$. Since $A > 0$, there exists a k such that $f(k) > g'(k)$. Define

$$g(i) = \begin{cases} g'(i) & \text{if } i \neq k \\ g'(i)+1 & \text{if } i = k. \end{cases}$$

Then $A(f,g) = A$. We calculate $\sum_i i \cdot (g(i) - g(i+1)) = \sum_i i \cdot (g'(i) - g'(i+1)) + 1$. By induction hypothesis, we have $\sum_i i \cdot (f(i) - f(i+1)) \ge \sum_i i \cdot (g(i) - g(i+1)) > \sum_i i \cdot (g'(i) - g'(i+1))$. □

By [19] the filtrations of the tangent bundle on $\mathcal{T}_X$ on a smooth toric variety X have the following form.

$$E^\rho(j) = \begin{cases} N \otimes k & j < 1 \\ \operatorname{span}_k(v_\rho) & j = 1 \\ 0 & j > 1 \end{cases} \tag{3}$$

Proof of Proposition 1.2 Looking at the filtrations $E^\rho(i)$ for $\mathcal{T}_X$ from (3) we see that

$$e^{[\rho]}(i) = E^\rho(i) - E^\rho(i+1) = \begin{cases} n-1 & j = 0 \\ 1 & j = 1 \\ 0 & \text{else.} \end{cases}$$

Similarly, for a proper subspace $F \subset N_{\mathbb{R}}$ and $F^\rho(i) = E^\rho(i) \cap F$, we have

$$f^{[\rho]}(i) = \begin{cases} \dim(F) - 2 & j = 0 \\ 1 & j = 1 \\ 0 & \text{else,} \end{cases} \quad \text{or} \quad f^{[\rho]}(i) = \begin{cases} \dim(F) - 1 & j = 0 \\ 0 & \text{else,} \end{cases}$$

depending on whether v_ρ is contained in the subspace F or not. Now, Proposition 2.3 immediately implies that $\mathcal{T}_X$ is (semi-)stable if and only if (1) holds for every proper subspace $F \subset N_{\mathbb{R}}$. To see that it suffices to test (1) for subspaces of the form $F = \operatorname{span}_k R$ with $R \subset \Sigma(1)$, assume that $\mathcal{F}$, given by some $F \subset N \otimes k$, destabilises $\mathcal{T}_X$. Then we may choose $\mathcal{F}'$ corresponding to $F' := \operatorname{span}\{v_\rho \subset \Sigma(1) \mid v_\rho \subset F\}$. With this choice we have $\sum_{v_\rho \in F} \operatorname{vol}(P^\rho) = \sum_{v_\rho \in F'} \operatorname{vol}(P^\rho)$ and $\operatorname{rk} \mathcal{F}' = \dim F' \le \dim F = \operatorname{rk} \mathcal{F}$. □

Example 2.6 For $\mathbb{P}^n$ a polarisation is given by $\mathcal{O}(d)$. The corresponding polytope is a d-fold dilation of the standard simplex $\Delta \subset \mathbb{R}^n$. Every facet of $d\Delta$ has lattice volume d^{n-1}. For every proper subset $R \subsetneq \Sigma(1)$ and

$F = \text{span}\, R$ we have $\dim F = \#R$. Now (1) becomes $d^{n-1} < d^{n-1} \cdot (n+1)/n$. Hence, we recover the well-known fact that $\mathbb{P}^n$ has a stable tangent bundle.

Lemma 2.7 *Assume that X is $\mathbb{Q}$-factorial and* $\text{Stab}(\mathcal{T}_X)$ *is non-empty. If there is a birational morphism $f: X' \to X$, then* $\text{Stab}(\mathcal{T}_{X'})$ *is non-empty, as well.*

Proof Consider a polarisation $\mathcal{O}(D)$ of X, such that $\mathcal{T}_X$ is stable. Then $\mathcal{T}_{X'}$ is stable with respect to the nef and big bundle $\mathcal{O}(f^*D)$, since any destabilising subsheaf $\mathcal{F}' \subset \mathcal{T}_{X'}$ with respect to $\mathcal{O}(f^*D)$ would induce a subsheaf $(f_*\mathcal{F}') \subset \mathcal{T}_X$ which, by projection formula, would be destabilising with respect to $\mathcal{O}(D)$. Now, the openness property from [14, Thm 3.3] ensures the existence of a stabilising ample class, which is given as a small pertubation of $[\mathcal{O}(f^*D)]$. □

We also have the following equivalent for the strictly unstable case.

Lemma 2.8 *Assume that X is $\mathbb{Q}$-factorial and* $\text{Amp}(X) \setminus \text{sStab}(\mathcal{T}_X)$ *is non-empty. If there is a birational morphism $f: X' \to X$, then* $\text{Amp}(X') \setminus \text{sStab}(\mathcal{T}_{X'})$ *is also non-empty.*

Proof Assume that a subsheaf $\mathcal{F} \subset \mathcal{T}_X$ destabilises $\mathcal{T}_X$ strictly with respect to an ample polarisation $\mathcal{O}(D)$. Then we note that $f^*\mathcal{F}$ and $\mathcal{T}_{X'}$ are both subsheaves of $f^*\mathcal{T}_X$. Now, we claim that $\mathcal{F}' := f^*\mathcal{F} \cap \mathcal{T}_{X'}$ destabilises $\mathcal{T}_{X'}$ with respect to $\mathcal{O}(D') = f^*\mathcal{O}(D)$. Indeed, by the projection formula we obtain

$$\begin{aligned} c_1(\mathcal{T}_{X'}).(D')^{n-1} &= c_1(\mathcal{T}_X).(D)^{n-1} \\ c_1(\mathcal{F}').(D')^{n-1} &= c_1(\mathcal{F}).(D)^{n-1}. \end{aligned}$$

The line bundle $\mathcal{O}(D')$ is only nef, but the condition that a subsheaf destabilises strictly is an open condition on the divisor class. Hence, we can find an ample divisor class with the same property as a small perturbation of $\mathcal{O}(D')$. □

3 Smooth toric surfaces

Every toric surface can be obtained via equivariant blow-ups from $\mathbb{P}^2$ or from a Hirzebruch surface $F_a = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a))$, see e.g. [28]. For $\mathbb{P}^2$ it is well-known that the tangent bundle is stable (see also Example 2.6). The following corollary, which can be also found e.g. in [2, Sec. 6], clarifies the situation for the Hirzebruch surfaces.

Corollary 3.1 *For a Hirzebruch surface $F_a = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a))$ the tangent bundle is semistable with respect to $\mathcal{O}_{F_a}(\lambda) \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(\mu)$ in the following cases*

1 $a = 0$ *and* $\lambda = \mu$,
2 $a = 1$ *and* $2\mu \leq \lambda$.

The tangent bundle is stable if and only if $a = 1$ *and* $2\mu < \lambda$.

Proof The claim follows directly from Theorem 1.4 for the case $r = s = 1$. □

Lemma 3.2 *A smooth toric surface* $X = X_\Sigma$ *is not a blowup of* $\mathbb{P}^2$ *or* $\mathbb{P}^1 \times \mathbb{P}^1$ *if and only if there are integers* a, c, e *fulfilling* $a \geq c > e + 1 \geq 1$ *such that after an appropriate choice of basis for* N:

1 Σ *contains the rays spanned by*

$$(0,1), (1,0), \quad (0,-1), (1,-e), \quad (-1,c), (-1,a)$$

2 *all other rays are contained in the cones* $\langle(-1,c),(-1,a)\rangle$ *and* $\langle(1,0),(1,-e)\rangle$.

Remark 3.3 Note, that in Lemma 3.2 we explicitly allow the cases $(1,-e) = (1,0)$ $(-1,c) = (-1,a)$.

Proof For example by [28, Thm. 1.28] we may find a ray $\langle v\rangle \in \Sigma(1)$ such that $-v$ spans another ray in $\Sigma(1)$. We then may number the ray generators

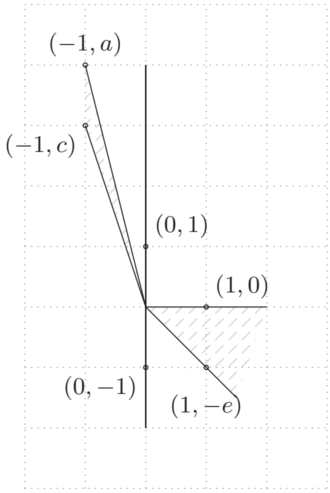

Figure 1 Schematic picture of a fan of a toric surface which does not blow down to $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. All additional rays have to be contained in the shaded regions.