CAMBRIDGE

Cambridge University Press
978-1-108-79250-9 — Facets of Algebraic Geometry
Edited by Paolo Aluffi , David Anderson , Milena Hering , Mircea Mustaţă , Sam Payne
Excerpt
More Information

1

Positivity of Segre–MacPherson Classes

Paolo Aluffi[a], Leonardo C. Mihalcea[b], Jörg Schürmann[c] and Changjian Su[d]

Dedicated to William Fulton on the occasion of his 80th birthday

Abstract. Let X be a complex nonsingular variety with globally generated tangent bundle. We prove that the signed Segre–MacPherson (SM) class of a constructible function on X with effective characteristic cycle is effective. This observation has a surprising number of applications to positivity questions in classical situations, unifying previous results in the literature and yielding several new results. We survey a selection of such results in this paper. For example, we prove general effectivity results for SM classes of subvarieties which admit proper (semi-)small resolutions and for regular or affine embeddings. Among these, we mention the effectivity of (signed) Segre–Milnor classes of complete intersections if X is projective and an alternation property for SM classes of Schubert cells in flag manifolds; the latter result proves and generalizes a variant of a conjecture of Fehér and Rimányi. Among other applications we prove the positivity of Behrend's Donaldson–Thomas invariant for a closed subvariety of an abelian variety and the signed-effectivity of the intersection homology Chern class of the theta divisor of a non-hyperelliptic curve; and we extend the (known) non-negativity of the Euler characteristic of perverse sheaves on a semi-abelian variety to more general varieties dominating an abelian variety.

1 Introduction

In this note, X will denote a nonsingular complex variety and $Z \subseteq X$ will be a closed subvariety; here (sub)varieties are by definition irreducible and reduced. We will assume that the tangent bundle of X is globally generated. In the

[a] Florida State University
[b] Virginia Tech University
[c] Universität Münster
[d] University of Toronto

© in this web service Cambridge University Press www.cambridge.org

Cambridge University Press
978-1-108-79250-9 — Facets of Algebraic Geometry
Edited by Paolo Aluffi , David Anderson , Milena Hering , Mircea Mustaţă , Sam Payne
Excerpt
More Information

projective case, this is equivalent to asking that X be a projective homogeneous variety – for example a projective space, a flag manifold, or an abelian variety; but our main results will hold in the non-complete case as well. We denote by $A_*(Z)$ the Chow group of cycles on Z modulo rational equivalence, and by $\mathsf{F}(Z)$ the group of constructible functions on Z; here we allow Z to be more generally a closed reduced subscheme of X. Our general aim is to investigate the *positivity* of certain rational equivalence classes associated with the embedding of Z in X, or more generally with suitable constructible functions on Z. We generalize several known positivity results on e.g., Euler characteristics, and provide a framework leading to analogous results in a broad range of situations.

Answering a conjecture of Deligne and Grothendieck, MacPherson [35] constructed a group homomorphism $c_*\colon \mathsf{F}(Z) \to A_*(Z)$ which commutes with proper push-forwards and satisfies a normalization property: if Z is nonsingular, then $c_*(1\!\!1_Z) = c(TZ) \cap [Z]$, where $c(TZ)$ is the total Chern class of Z. (MacPherson worked in homology; see [25, Example 19.1.7] for the refinement of the theory to the Chow group.) If $Y \subseteq Z$ is a constructible subset, the Chern–Schwartz–MacPherson (CSM) class $c_{\mathrm{SM}}(Y) \in A_*(Z)$ is the image $c_*(1\!\!1_Y)$ of the indicator function of Y under MacPherson's natural transformation. Let $\varphi \in \mathsf{F}(Z)$. We will focus on the closely related Segre–MacPherson (SM) class

$$s_*(\varphi, X) := c(TX|_Z)^{-1} \cap c_*(\varphi) \in A_*(Z).$$

(The class $c(TX|_Z)$ is invertible in $A_*(Z)$, because it is of the form $1 + a$, where a is nilpotent.) In particular, we let $s_{\mathrm{SM}}(Y, X)$ denote the Segre-Schwartz-MacPherson (SSM) class $s_*(1\!\!1_Y, X) \in A_*(Z)$; note that this class depends on both Y and the ambient variety X. If Y is a subvariety of Z, then the top-degree component of $s_{\mathrm{SM}}(Y, X)$ in $A_{\dim Y}(Z)$ is the fundamental class $[\overline{Y}]$ of the closure of Y. Further, if $Y = Z$ is a *nonsingular* closed subvariety of X, then $s_{\mathrm{SM}}(Y, X) \in A_*Y$ equals the ordinary Segre class $s(Y, X)$; in general, the two classes differ (as discussed in subsection 8.2 for Y a global complete intersection in a nonsingular projective variety X). See [1] and (in the equivariant case) [37] for general properties of SM classes, and [43] for their compability with transversal pullbacks.

There are 'signed' versions of both c_* and s_* (respectively, c_{SM} and s_{SM}), which appear naturally when relating them to characteristic cycles. If $c_*(\varphi) = c_0 + c_1 + \cdots$ is the decomposition into homogeneous components (i.e., $c_i \in A_i(Z)$) then the 'signed' class $\check{c}_*(\varphi)$ is defined by

$$\check{c}_*(\varphi) = c_0 - c_1 + c_2 - \cdots \quad \text{and} \quad \check{c}_{\mathrm{SM}}(Y) := \check{c}_*(1\!\!1_Y);$$

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE
Cambridge University Press
978-1-108-79250-9 — Facets of Algebraic Geometry
Edited by Paolo Aluffi , David Anderson , Milena Hering , Mircea Mustaţă , Sam Payne
Excerpt
More Information

that is, by changing the sign of each homogeneous component of odd dimension. One defines similarly the signed SM class

$$\check{s}_*(\varphi, X) := c(T^*X|_Z)^{-1} \cap \check{c}_*(\varphi) \quad \text{and} \quad \check{s}_{\mathrm{SM}}(Y, X) := \check{s}_*(1\!\!1_Y, X).$$

A basis for $\mathsf{F}(Z)$ consists of the *local Euler obstructions* Eu_Y for closed subvarieties Y of Z. In fact, the characteristic cycle of the (signed) local Euler obstruction is an irreducible Lagrangian cycle in T^*X, and from this perspective the functions Eu_Y are the 'atoms' of the theory; see equation (2) below. If Y is nonsingular, $\mathrm{Eu}_Y = 1\!\!1_Y$. The local Euler obstruction is a subtle and well-studied invariant of singularities (see e.g., [35, 19, 31, 21, 13, 7]). The corresponding class $c_*(\mathrm{Eu}_Z) \in A_*(Z)$ for Z a subvariety of X is the *Chern–Mather class* of Z, $c_{\mathrm{Ma}}(Z) = \nu_*(c(\tilde{T}) \cap [\tilde{Z}])$, with $\nu\colon \tilde{Z} \to Z$ the *Nash blow-up* of Z and $\tilde{T}$ the tautological bundle on $\tilde{Z}$ extending TZ_{reg}, cf. [35] or [25, Example 4.2.9]. In particular $c_{\mathrm{Ma}}(Z) = c(TZ) \cap [Z]$ if Z is nonsingular. If Z is complete, we denote by $\chi_{\mathrm{Ma}}(Z) := \chi(Z, \mathrm{Eu}_Z)$ the degree of $c_{\mathrm{Ma}}(Z)$; so $\chi_{\mathrm{Ma}}(Z)$ equals the usual topological Euler characteristic $\chi(Z)$ if Z is nonsingular and complete. We also consider the corresponding Segre–Mather class $s_{\mathrm{Ma}}(Z, X) := s_*(\mathrm{Eu}_Z, X)$ as well as the signed classes

$$\check{c}_{\mathrm{Ma}}(Z) := \check{c}_*(\mathrm{Eu}_Z); \quad \check{s}_{\mathrm{Ma}}(Z, X) := c(T^*X|_Z)^{-1} \cap \check{c}_*(\mathrm{Eu}_Z).$$

With our conventions, we get $(-1)^{\dim Z}\check{c}_{\mathrm{Ma}}(Z) = \nu_*(c(\tilde{T}^*) \cap [\tilde{Z}])$ in terms of the dual tautological bundle on the Nash blow-up (which differs by the sign $(-1)^{\dim Z}$ from the definition of the signed Chern–Mather class used in some references like [40, 44]).

The main result in this paper is the (signed) effectivity of Segre–MacPherson classes in a large class of examples. By an *effective* class we mean a class which can be represented by a nonzero, non-negative, cycle.

Theorem 1.1 *Let X be a complex nonsingular variety, and assume that the tangent bundle TX is globally generated. Let $Z \subseteq X$ be a closed subvariety of X. Then the following hold:*

(a) The class $(-1)^{\dim Z}\check{s}_{Ma}(Z, X) \in A_*(Z)$ is effective.

(b) Assume that the inclusion $U \hookrightarrow Z$ is an affine morphism, where U is a locally closed smooth subvariety of Z. Then $(-1)^{\dim U}\check{s}_{SM}(U, X) \in A_*(Z)$ is effective.

If in addition X is assumed to be complete, then the requirement that TX is globally generated is equivalent to X being a homogeneous variety; cf. e.g., [17, Corollary 2.2]. Further, Borel and Remmert [11] (see also [17, Theorem 2.6]) prove that all complete homogeneous varieties are products

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-108-79250-9 — Facets of Algebraic Geometry
Edited by Paolo Aluffi , David Anderson , Milena Hering , Mircea Mustaţă , Sam Payne
Excerpt
More Information

$(G/P) \times A$, where G is a semisimple Lie group, $P \subseteq G$ is a parabolic subgroup, and A is an abelian variety.

Theorem 1.1 is extended to more general constructible functions in Theorem 2.2 below. These theorems may be used to prove several positivity statements, unifying and generalizing analogous results from the existing literature. We list below the situations which we will highlight in this paper to illustrate applications of our methods, and the sections where these are discussed.

(a) Closed subvarieties of abelian varieties; primarily in §3.
(b) The proof of a generalization of a conjecture of Fehér and Rimányi [22] concerning SSM classes of Schubert cells in Grassmannians; §4.1.
(c) Complements of hyperplane arrangements; §4.2
(d) Positivity of certain Donaldson–Thomas type invariants; §5.
(e) Intersection homology Segre and Chern classes; §6.
(f) Semi-small resolutions; §8.1.
(g) Regular embeddings and Milnor classes; §8.2.
(h) Semi-abelian varieties and generalizations; §8.3.

Ultimately, these positivity statements follow from the effectivity of the associated characteristic cycles. In §7 we survey a more comprehensive list of situations in which the characteristic cycle is positive.

The proof of Theorem 1.1 will be given in §2.2 below. It is suprisingly easy, but not elementary; it is based on a classical formula by Sabbah [40], calculating the (signed) CSM class of a constructible function φ in terms of its characteristic cycle $\mathrm{CC}(\varphi)$; see Theorem 2.1 below.

Acknowledgements. P. Aluffi acknowledges support from the Simons Collaboration Grant 625561; L. C. Mihalcea was supported in part by NSA Young Investigator Award H98320-16-1-0013 and the Simons Collaboration Grant 581675; J. Schürmann was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy – EXC 2044–390685587, Mathematics Münster: Dynamics – Geometry – Structure.

P.A. thanks the University of Toronto for the hospitality. L.M. is grateful to R. Rimányi for many stimulating discussions about the CSM and SM classes, including the positivity conjecture from [22], and to the Math Department at UNC Chapel Hill for the hospitality during a sabbatical leave in the academic year 2017–18. The authors are grateful to an anonymous referee whose comments prompted a reorganization of this paper.

© in this web service Cambridge University Press www.cambridge.org

Cambridge University Press
978-1-108-79250-9 — Facets of Algebraic Geometry
Edited by Paolo Aluffi , David Anderson , Milena Hering , Mircea Mustaţă , Sam Payne
Excerpt
More Information

Finally, this paper is dedicated to Professor William Fulton in the occasion of his 80th birthday. His interest in positivity questions arising in algebraic geometry, and his influential ideas, continue to inspire us.

2 Characteristic Classes via Characteristic Cycles; Proof of the Main Theorem

2.1 Characteristic Cycles

Let X be a smooth complex variety. We recall a commutative diagram which plays a central role in seminal work of Ginzburg [27]; it is largely based on results from [8, 18, 30]. We also considered this diagram in our previous work [5, §4.2], and we use the notation from this reference.

$$\begin{array}{ccc} \mathrm{Perv}(X) & \xleftarrow[\sim]{\mathrm{DR}} & \mathrm{Mod}_{rh}(\mathcal{D}_X) \\ {\scriptstyle \chi_{stalk}}\downarrow & & \downarrow{\scriptstyle \mathrm{Char}} \\ \mathsf{F}(X) & \xrightarrow[\sim]{\mathrm{CC}} & L(X) \end{array} \tag{1}$$

Here $\mathrm{Mod}_{rh}(\mathcal{D}_X)$ denotes the abelian category of algebraic holonomic $\mathcal{D}_X$-modules with regular singularities, and $\mathrm{Perv}(X)$ is the abelian category of perverse (algebraically) constructible complexes of sheaves of $\mathbb{C}$-vector spaces on X; $\mathsf{F}(X)$ is the group of constructible functions on X and $L(X)$ is the group of conic Lagrangian cycles in T^*X. The functor DR is defined on $M \in \mathrm{Mod}_{rh}(\mathcal{D}_X)$ by

$$\mathrm{DR}(M) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, M)[\dim X],$$

that is, it computes the DeRham complex of a holonomic module (up to a shift), viewed as an *analytic* $\mathcal{D}_X$-module. This functor realizes the Riemann-Hilbert correspondence, and is an equivalence. We refer to e.g., [30, 27] for details. The left map χ_{stalk} computes the stalkwise Euler characteristic of a constructible complex, and the right map Char gives the characteristic cycle of a holonomic $\mathcal{D}_X$-module. The map CC is the characteristic cycle map for constructible functions; if $Z \subseteq X$ is closed and irreducible, then

$$\mathrm{CC}(\mathrm{Eu}_Z) = (-1)^{\dim Z}[T^*_Z X]; \tag{2}$$

here Eu_Z is the local Euler obstruction (see §1), and $T^*_Z X := \overline{T^*_{Z^{reg}} X}$ is the conormal space of Z, i.e., the closure of the conormal bundle of the smooth

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-108-79250-9 — Facets of Algebraic Geometry
Edited by Paolo Aluffi , David Anderson , Milena Hering , Mircea Mustaţă , Sam Payne
Excerpt
More Information

locus of Z. The commutativity of diagram (1) is shown in [27] using deep $\mathcal{D}$-module techniques; it also follows from [41, Example 5.3.4, p. 359–360] (even for a holonomic $\mathcal{D}$-module without the regularity requirement). Also note that the upper transformations in (1) factor over the corresponding Grothendieck groups, so they also apply to complexes of such $\mathcal{D}$-modules. If $f: X \to Y$ is a proper map of smooth complex varieties, there are well-defined push-forwards for each of the objects in the diagram, denoted by f_*. Furthermore, all the maps commute with proper push forwards; cf. [27, Appendix]. For other proofs, see [29, Proposition 4.7.5] for the transformation DR, [41, §2.3] for the transformation χ_{stalk} and [42, §4.6] for the transformation CC (for the transformation Char it then follows from the commutativity of diagram (1)).

The next result, relating characteristic cycles to (signed) CSM classes, has a long history. See [40, Lemme 1.2.1], and more recently [38, (12)], [42, §4.5], [43, §3], especially diagram (3.1) in [43].

Theorem 2.1 *Let X be a complex nonsingular variety, and let $Z \subseteq X$ be a closed reduced subscheme. Let $\varphi \in \mathsf{F}(Z)$ be a constructible function on Z. Then*

$$\check{c}_*(\varphi) = c(T^*X|_Z) \cap \mathrm{Segre}(\mathrm{CC}(\varphi))$$

as elements in the Chow group $A_*(Z)$ of Z. Here $\mathrm{Segre}(\mathrm{CC}(\varphi))$ *is the Segre class associated to the conic Lagrangian cycle* $\mathrm{CC}(\varphi) \subseteq T^*X|_Z$.

We recall the definition of the Segre class used in Theorem 2.1. Let $q: \mathbb{P}(T^*X|_Z \oplus 1\!\!1) \to Z$ be the projection from the restriction of the projective completion of the cotangent bundle of X. If $C \subseteq T^*X|_Z$ is a cone supported over Z, and $\overline{C}$ is the closure in $\mathbb{P}(T^*X|_Z \oplus 1\!\!1)$, the Segre class is defined by

$$\mathrm{Segre}(C) := q_*\left(\sum_{i\geq 0} c_1\left(\mathcal{O}_{\mathbb{P}(T^*X|_Z\oplus 1\!\!1)}(1)\right)^i \cap [\overline{C}]\right)$$

as an element of $A_*(Z)$; see [25, §4.1].

Every irreducible conic Lagrangian subvariety of T^*X is a conormal cycle $T^*_Z X$ for $Z \subseteq X$ a closed subvariety; see e.g., [29, Theorem E.3.6]. From this it follows that every non-trivial characteristic cycle is a linear combination of conormal spaces:

$$\mathrm{CC}(\varphi) = \sum_Y a_Y [T^*_Y X] \qquad (3)$$

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE
Cambridge University Press
978-1-108-79250-9 — Facets of Algebraic Geometry
Edited by Paolo Aluffi , David Anderson , Milena Hering , Mircea Mustaţă , Sam Payne
Excerpt
More Information

for uniquely determined closed subvarieties Y of Z and nonzero integer coefficients a_Y. By (2), the coefficients a_Y are determined by the equality of constructible functions

$$0 \neq \varphi = \sum_Y a_Y (-1)^{\dim Y} \mathrm{Eu}_Y. \tag{4}$$

2.2 Proof of the Main Theorem

The following result is at the root of all applications in this note.

Theorem 2.2 *Let X be a complex nonsingular variety such that TX is globally generated. Let $Z \subseteq X$ be a closed reduced subscheme of X and let $\varphi \in \mathsf{F}(Z)$ be a constructible function on Z such that the characteristic cycle $\mathrm{CC}(\varphi) \in A_*(T^*X|_Z)$ is effective.*

Then $\check{s}_*(\varphi, X)$ is effective in $A_*(Z)$. If TX is trivial (e.g., X is an abelian variety), then $\check{c}_*(\varphi)$ is effective.

Proof By Theorem 2.1,

$$\check{s}_*(\varphi, X) = c(T^*X|_Z)^{-1} \cap \check{c}_*(\varphi) = \mathrm{Segre}(\mathrm{CC}(\varphi)).$$

By hypothesis we have a decomposition (3) with positive coefficients a_Y. It follows from (4) that the Segre class of $\mathrm{CC}(\varphi)$ is a linear combination of Segre–Mather classes of subvarieties:

$$\begin{aligned} \mathrm{Segre}(\mathrm{CC}(\varphi)) &= \sum_Y a_Y (-1)^{\dim Y} \mathrm{Segre}(\mathrm{CC}(\mathrm{Eu}_Y)) \\ &= \sum_Y a_Y (-1)^{\dim Y} \check{s}_{\mathrm{Ma}}(Y, X). \end{aligned}$$

By definition, the top degree part of each signed Segre–Mather class $(-1)^{\dim Y} \check{s}_{\mathrm{Ma}}(Y, X)$ equals $[Y]$. Then the top degree part of $\mathrm{Segre}(\mathrm{CC}(\varphi))$ equals a positive linear combination of those fundamental classes $[Y]$ of maximal dimension, and in particular $\check{s}_*(\varphi, X) = \mathrm{Segre}(\mathrm{CC}(\varphi))$ is not zero. Since the tangent bundle TX is globally generated, it follows that the line bundle $\mathcal{O}_{\mathbb{P}(T^*X \oplus \mathbb{1})}(1)$ is globally generated, as it is a quotient of $TX \oplus \mathbb{1}$. Therefore its first Chern class preserves non-negative classes. Since non-negativity is preserved by proper push-forwards, we can conclude that under the given hypotheses $\mathrm{Segre}(\mathrm{CC}(\varphi))$ is non-negative, and this completes the proof. □

© in this web service Cambridge University Press www.cambridge.org

Cambridge University Press
978-1-108-79250-9 — Facets of Algebraic Geometry
Edited by Paolo Aluffi , David Anderson , Milena Hering , Mircea Mustaţă , Sam Payne
Excerpt
More Information

Theorem 1.1 follows from Theorem 2.2:

Proof of Theorem 1.1 By Theorem 2.2, it suffices to show that the characteristic cycles for the constructible functions $(-1)^{\dim Z}\mathrm{Eu}_Z$ and $(-1)^{\dim U}\mathbb{1}_U$ are effective. If $\varphi = \check{\mathrm{E}}\mathrm{u}_Z := (-1)^{\dim Z}\mathrm{Eu}_Z$, then $\mathrm{CC}(\varphi)$ equals the conormal cycle $[T^*_Z X]$ of Z, and it is therefore trivially effective.

Consider then $(-1)^{\dim U}\mathbb{1}_U$, and let $j: U \hookrightarrow X$ be the inclusion. We use the Riemann–Hilbert correspondence (diagram (1)) to express the characteristic cycle. By definition,

$$(-1)^{\dim U}\mathbb{1}_U = \chi_{stalk}(j_!\mathbb{C}_U[\dim U]).$$

Since U is nonsingular, the sheaf $\mathbb{C}_U[\dim U]$ is perverse, and $\mathcal{O}_U$ is the corresponding regular holonomic $\mathcal{D}_U$-module. We have $j_!\mathbb{C}_U[\dim U] = \mathrm{DR}(j_!(\mathcal{O}_U))$; since j is an affine morphism, $j_!(\mathcal{O}_U)$ is a single regular holonomic $\mathcal{D}_X$-module (with support in Z); see [29, p. 95]. As pointed out in [29, p. 119], the characteristic cycles of non-trivial holonomic $\mathcal{D}_X$-modules are effective, and this finishes the proof. □

Remark 2.3 As the proof shows, the hypothesis that U is smooth in Theorem 1.1(ii) can be weakened, by only requiring that that $\mathbb{C}_U[\dim U]$ is a perverse sheaf. The proof of the effectivity then uses the fact that for an affine inclusion $j: U \hookrightarrow Z$, $j_!\mathbb{C}_U[\dim U]$ is a perverse sheaf on Z ([41, Lemma 6.0.2, p. 384 and Theorem 6.0.4, p. 409]). We will formalize this conclusion below, in Proposition 2.4. For the case in which $U = X \smallsetminus D$ is the open complement of a hypersurface D in $Z := X$, the result also follows from [41, Proposition 6.0.2, p. 404]. ⌟

2.3 Effective Characteristic Cycles (I)

The applications in the rest of the paper follow from Theorem 2.2: they represent situations when the characteristic cycle $\mathrm{CC}(\varphi)$ is effective.

As pointed out above, every non-trivial characteristic cycle is a linear combination of conormal spaces (3), and the coefficients a_Y in a linear combination are determined by the equality of constructible functions (4). The characteristic cycle of $\varphi \neq 0$ is effective if and only if the coefficients a_Y are positive. In particular, this condition is intrinsic to the constructible function $\varphi \in \mathsf{F}(Z)$ and does not depend on the chosen closed embedding of Z into an ambient nonsingular variety X.

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-108-79250-9 — Facets of Algebraic Geometry
Edited by Paolo Aluffi , David Anderson , Milena Hering , Mircea Mustaţă , Sam Payne
Excerpt
More Information

A key source of examples where $\mathrm{CC}(\varphi)$ is effective, but possibly reducible, arises as follows. Constructible functions may be associated with (regular) holonomic $\mathcal{D}$-modules and perverse sheaves $\mathcal{F} \in \mathrm{Perv}(Z)$, cf. diagram (1); for example, in the latter case the value of the constructible function $\varphi := \chi_{stalk}(\mathcal{F})$ at the point $z \in Z$ is the Euler characteristic $\varphi(z) = \chi(\mathcal{F}_z)$ of the stalk at z of the given complex of sheaves $\mathcal{F}$.

Proposition 2.4 *Let X be a complex nonsingular variety such that TX is globally generated, and let $Z \subseteq X$ be a closed reduced subscheme. Let $0 \neq \varphi \in \mathsf{F}(Z)$ be a non-trivial constructible function associated with a regular holonomic $\mathcal{D}_X$-module supported on Z, or (equivalently) a perverse sheaf on Z. Then $\check{s}_*(\varphi, X)$ is effective in $A_*(Z)$.*

Proof This follows from the argument used in the proof of Theorem 2.2: the main observation is that the characteristic cycle of a non-trivial (regular) holonomic $\mathcal{D}$-module is effective; see e.g., [29, p. 119]. Further, perverse sheaves correspond to regular holonomic $\mathcal{D}$-modules by means of the Riemann–Hilbert correspondence (see e.g., [29, Theorem 7.2.5]), compatibly with the construction of the associated constructible functions and characteristic cycles; cf. diagram (1). □

There are situations where the characteristic cycle associated to a constructible sheaf is known to be irreducible: examples include characteristic cycles of the intersection cohomology sheaves of Schubert varieties in the Grassmannian [15], in more general minuscule spaces [10], of certain determinantal varieties [48], and of the theta divisors in the Jacobian of a non-hyperelliptic curve [14]. In all such cases, $\check{s}_*(\varphi, X)$ is effective provided that TX is globally generated, by Theorem 2.2. Also note that for the varieties Z listed above, the Chern–Mather class $c_{\mathrm{Ma}}(Z)$ equals $c_{\mathrm{IH}}(Z)$, the intersection homology class defined in §6 below. This follows because in this case the characteristic cycle of $\mathcal{IC}_Z$ is irreducible, thus it must agree with the conormal cycle of Z.

In the next few sections we discuss specific applications of Theorem 2.2 for various choices of the variety X or constructible function φ. The sections are mostly logically independent of each other, and the reader may skip directly to the case of interest. The only exception are the results concerning Abelian varieties; these will be mentioned throughout this note.

A more detailed discussion on effective characteristic cycles is given in §7 below, including a more comprehensive list of constructible functions φ for

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE
Cambridge University Press
978-1-108-79250-9 — Facets of Algebraic Geometry
Edited by Paolo Aluffi , David Anderson , Milena Hering , Mircea Mustaţă , Sam Payne
Excerpt
More Information

which $\mathrm{CC}(\varphi)$ is effective, and operations on characteristic functions which preserve the effectivity of the corresponding characteristic cycles.

3 Abelian Varieties

If X is an abelian variety, then TX is trivial. (In fact, this characterizes abelian varieties among complete varieties, cf. [17, Corollary 2.3].) If TX is trivial, then for all constructible functions φ on Z the signed SM class agrees with the signed CM class: $\check{s}_*(\varphi, X) = \check{c}_*(\varphi) \in A_*(Z)$. In particular, $\check{s}_{\mathrm{Ma}}(Z, X) = \check{c}_{\mathrm{Ma}}(Z)$ for Z a subvariety of X. The following result follows then immediately from Theorems 1.1 and 2.2.

Corollary 3.1 *Let Z be a closed subvariety of a smooth variety X with TX trivial (for example, an abelian variety). Then $(-1)^{\dim Z}\check{c}_{Ma}(Z)$ is effective.*

More generally, let φ be a constructible function on Z such that $\mathrm{CC}(\varphi)$ is effective. Then $\check{c}_*(\varphi) \in A_*(Z)$ is effective.

As an example, the total Chern–Mather class $c_{\mathrm{Ma}}(\Theta)$ of the theta divisor in the Jacobian of a nonsingular curve must be signed-effective.

Corollary 3.1 implies that $\chi(Z, \varphi) \geq 0$, which also follows from [24, Theorem 1.3]. In particular, if Z is a closed subvariety of an abelian variety, then

$$(-1)^{\dim Z}\chi_{\mathrm{Ma}}(Z) = (-1)^{\dim Z}\chi(Z, \mathrm{Eu}_Z) \geq 0.$$

For nonsingular subvarieties Z, the Euler obstruction Eu_Z equals $1\!\!1_Z$. Then the fact that $(-1)^{\dim Z}\chi(Z) \geq 0$ is proven (in the more general semi-abelian case) in [24, Corollary 1.5] (also see [20, (2)]). We note that the fact that $c(T^*Z) \cap [Z]$ is effective if Z is a *nonsingular* subvariety of an abelian variety X also follows immediately from the fact that T^*Z is globally generated, as it is a homomorphic image of the restriction of T^*X, which is trivial. Corollary 3.1 extends this result to *arbitrarily singular* closed subvarieties of a smooth variety X with trivial tangent bundle.

In fact, Corollary 3.1 also follows from Propositions 2.7 and 2.9 from [44], where explicit effective cycles representing $\check{s}_*(\varphi, X)$ in terms of suitable 'polar classes' are constructed.

4 Affine Embeddings

An important family of positivity statements arises from the indicator function $1\!\!1_U$ of a typically nonsingular and noncompact subvariety U of $Z \subseteq X$. In this

© in this web service Cambridge University Press www.cambridge.org