1 Introduction

Many complex systems consist of multiple interacting subsystems. Examples include financial (Caccioli et al., 2014; Huang et al., 2013), infrastructure (Rinaldi, Peerenboom, & Kelly, 2001), informatic (Leicht & D'Souza, 2009), and ecological (Pocock, Evans, & Memmott, 2012) systems. To understand the functioning of one subsystem or layer, it is necessary to take into account dependencies upon and interactions with the other subsystems, which can significantly alter the behaviour of the system (Buldyrev et al., 2010).

A convenient and powerful representation for many such systems is as a set of interdependent networks, with each subsystem represented by a separate network layer. Interdependencies are represented as special dependency links between nodes in different layers. The presence of a dependency link indicates that the failure of a node in one layer will lead to the failure of nodes in other layers to which it is connected by such dependency links. These interdependencies may dramatically increase the fragility of the system (Baxter et al., 2012; Buldyrev et al., 2010). The dependency may be full, so that every node in one layer is interdependent with a partner node in each other layer, or partial, so that some nodes have dependencies in only some of the other layers (Dong et al., 2012).

In many cases, a common set of nodes can be defined across all layers, allowing a multiplex network representation (Son et al., 2012), which simplifies the analysis. That is, if the dependency connections are one-to-one, we may merge interdependent nodes, as they will always be removed together. The different layers then consist of different types (colours) of connections between the same set of nodes. Such a network, one set of nodes with different types of edges between them, is called a multiplex network. Note that this mapping is still possible even when the multilayer network is only partially interdependent, or when not all nodes appear in every layer. Nodes which do not appear in a given layer have no connections of the corresponding colour in the multiplex representation.

In a single network, a giant connected component, containing a finite fraction of all nodes in the network, appears at a well-defined percolation threshold with respect to a control parameter affecting the density of the network (mean degree, or fraction of nodes or edges surviving random damage, for example). This transition is typically a second-order continuous transition, with the relative size of the giant component growing linearly immediately above the threshold in random networks, which is the mean-field scaling, although this growth exponent may be strongly affected if the degree distribution is

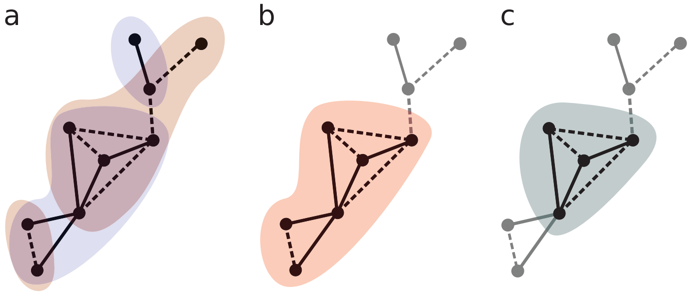

Figure 1 Comparison between percolation generalisations in multiplex networks. (a) A two-layer multiplex network, with connected clusters in each layer shaded. (b) A connected cluster under weak multiplex percolation. Nodes must have at least one connection of each type to at least one other member of the cluster. Nodes do not necessarily belong to the same cluster in each layer when considered separately. (c) A mutually connected cluster. Nodes must have a path to all other members of the cluster in both layers (i.e. they must belong to the same connected cluster in both layers).

very broad (Cohen, Ben-Avraham, & Havlin, 2002; Dorogovtsev, Goltsev, & Mendes, 2008).

The percolation problem may be generalised to multiplex or multilayer networks in two ways:

(i) In the percolation process described in Baxter et al. (2012); Buldyrev et al. (2010); Son et al. (2012), a percolating cluster consists of a set of nodes, each pair of which is connected by a path in every layer of the multiplex network to which they both belong; see Figure 1 (c). Such clusters are referred to as mutually connected clusters. One finds a greatly increased fragility of the system. Failures of nodes in one layer lead to failures in another layer, affecting the connectivity of other nodes in that layer, which may then fail, causing further failures in the first layer. In this way, damage may cascade back and forth between layers. A small initial damage may lead to a discontinuous collapse of the giant mutually connected cluster. The collapse is a discontinuous hybrid phase transition, which differs from a first-order transition in that one finds a square root singularity above the transition, with diverging susceptibility. The phase transition therefore has some properties in common with second-order transitions. The same type of transition is observed in *k*-core percolation (Dorogovtsev, Goltsev, & Mendes, 2006). A consequence of this definition is that in order to establish whether a given node belongs to a percolating cluster, one must explore the whole cluster to which it belongs, in all layers in which it is present. One may identify the giant mutually connected component by iteratively removing all finite connected components in each layer until an equilibrium is reached. Finding the giant mutually connected cluster is therefore a computationally intensive process, as compared with, say, k-core pruning (Baxter et al., 2015), in which a local pruning rule may be applied to find the giant k-core cluster.

Since its proposal by Buldyrev et al. (2010), significant attention has been devoted to this process, exploring the effects of partial interdependence (Dong et al., 2012), multiple dependencies (Shao et al., 2011), correlations (Hu et al., 2013), and overlapping edges (Baxter et al., 2016; Cellai, Dorogovtsev, & Bianconi, 2016; Min et al., 2015) among many others (Bianconi, 2018; Boccaletti et al., 2014; Cozzo et al., 2018; Kivelä et al., 2014). The mutually connected component in multiplex networks is strongly affected by highly heterogeneous network structure (Baxter et al., 2012). In two layers with powerlaw-tailed degree distributions, one finds that both the height of the discontinuity and the critical point tend to zero as the powerlaw exponent tend to $\gamma = 2$ from above.

(ii) An alternative percolation rule was proposed by Baxter, Dorogovtsev, Mendes, and Cellai (2014). Under this rule, nodes are considered active if they maintain at least one connection to another active node in each layer to which it belongs. Connected clusters are formed from such active nodes, with two nodes belonging to the same cluster if there is a path between them ignoring to which layer the edges belong. A node belongs to a given cluster if it is connected to at least one member of the cluster in every layer to which it belongs. There is not necessarily a path between every pair of nodes in the cluster in every layer, see Figure 1 (b). Under this rule, the giant component may be identified by iteratively applying a local pruning process, removing any nodes without the required connections, without needing to repeatedly identify the full clusters. The computation required is similar to that of k-core pruning. This problem is referred to as weak multiplex percolation to distinguish it from the more restrictive rule of the mutually connected cluster.

This process was further explored in Min and Goh (2014), and the relationship with the stronger rule elaborated in Baxter, Cellai, Dorogovtsev, Goltsev, & Mendes (2016). A complete description of the critical phenomena associated with weak multiplex percolation was given in Baxter, da Costa, Dorogovtsev, and Mendes (2020). In two-layer networks the problem is equivalent to (1−1)-core percolation, as proposed in Azimi-Tafreshi, Gómez-Gardenes, and Dorogovtsev (2014). One observes a continuous transition in two layers, but typically with quadratic (rather than linear) growth above the critical point. In three or more layers a discontinuous hybrid phase transition occurs, of the same type as for the mutually connected components (rule (i) above).

The choice of which multiplex percolation definition is appropriate depends on the problem in hand. In many situations, the functioning or survival of an agent or site is dependent only on its relations with close neighbours – support and distribution of distinct resources or information, for example, Min and Goh (2014). For such problems, weak multiplex percolation applies. For systems requiring connection to a common or centralised system, such as electricity supply and control, the mutually connected clusters might be more appropriate (Buldyrev et al., 2010). Connected components for these two percolation formulations are compared in Figure 1.

The unusual phase transitions observed in both these problems, a hybrid transition which is discontinuous like a first-order transition, and a square root singularity on one side of the transition, with diverging susceptibility like a second-order transition, have also been observed in k-core percolation (Dorogovtsev et al., 2006), in the activation process of bootstrap percolation (Baxter et al., 2010) and the Kuramoto synchronisation model (Moreno & Pacheco, 2004), as well as in activation processes on multiplex and multi-layer networks (Baxter et al., 2014; Min & Goh, 2014) among others. What these processes have in common is the possibility of avalanches. A change in the status of one node may alter the status of a neighbouring node, which in turn may affect further nodes. For example, in the k-core problem, nodes belong to the k-core if they have at least k neighbours within the core. If a node is removed, its neighbours lose a connection, and may then have less than k connections, so are themselves removed. These chains or avalanches of removals can be seen as a branching process. The phase transition occurs when the branching ratio exceeds one, so each removal, on average, leads to more than one further removal. The avalanches diverge in size and consume a finite fraction of the system, leading to a discontinuity in the system size (Baxter et al., 2015).

Our aim in this Element is to summarise the behaviour observed under weak multiplex percolation, the methods used to calculate the size of the giant component, and highlight the unusual critical phenomena observed in this problem. We detail the effects of heterogeneous degree distributions, showing that, in contrast with other network percolation problems, the discontinuity and critical point remain nonzero at $\gamma = 2$, finally vanishing at $\gamma = 1 + 1/(M - 1)$, where M is the number of layers. These phenomena are placed in context with related percolation problems. The results presented here are based largely on those presented in Baxter et al. (2014) and Baxter et al. (2020).

The remainder of this Element is organised as follows. In the Section 2 we precisely define the problem and give the self-consistency equations which allow its solution. We compare weak multiplex percolation with ordinary percolation and with the mutually connected component in Section 3. In Section 4 we derive the main results for rapidly decaying degree distributions. The effect of heterogeneous degree distributions is then detailed in Section 5, including the disappearance of the hybrid transition for very slowly decaying degree distributions. Final discussion is given in Section 6.

2 Weak Multiplex Percolation

For the purposes of this Element, we will assume a multiplex formulation of the problem. A multiplex network consists of a set of N nodes, connected in $M > 1$ layers, each with its own type (colour) of edge. A node may be connected in all M layers, or only a subset of them. In a simple network (i.e. $M = 1$ layer), two nodes belong to the same connected component if there is a path between them following the edges of the network. One may vary the density of connections in the network; for example, by changing the mean degree or occupying edges or nodes with probability p. When the network is very sparse, with few connections, connected components are small. As we increase the density of connections, connected clusters grow and may start to join together to form larger clusters. Eventually, the largest connected component may contain a significant fraction of all the nodes in the network. In the infinite size limit $N \to \infty$, connected components are either finite (containing a finite number of nodes) and thus occupy a vanishing fraction of the whole network, or giant, occupying a non-zero fraction of the network. Beginning with a low density, only finite clusters exist. As we increase the network density, a giant connected component appears at a well-defined threshold, above which the giant component grows linearly for sufficiently rapidly decaying degree distributions. This is the classic percolation transition.

In weak multiplex percolation, a node is active if it maintains a connection in each layer to other active nodes. Two active nodes are part of the same weak percolation component if there is a path between them via edges in any layer (i.e. ignoring edge colours). To identify the weak percolation clusters in a given multiplex network, one may simply prune any nodes that do not have connections in all layers (inactive nodes) in which they participate, repeating the pruning until an equilibrium is reached. The connected clusters in the projection of the remaining network are then the weak percolating clusters (Baxter et al., 2014; Baxter, Cellai, et al., 2016).

Identifying Weak Multiplex Percolation Clusters

In a given multiplex network, one may use the following algorithm to identify the weak percolating clusters:

1. Set the status of all nodes in the network to active.
2. For each node i, for each layer s in which i has degree > 0, check whether i has at least one active neighbour in s. If not, set the status of i to inactive.
3. Repeat step 2 until no changes to node statuses are made.
4. Identify connected clusters within the subnetwork consisting only of active nodes and the edges between them. These are the weak percolating clusters.

In a two-layer network with sufficiently rapidly decaying degree distribution (such as Poisson degree distributions, as found in Erdős–Rényi networks), the giant component appears continuously with a second-order phase transition and grows as the square of the distance from the critical point. This differs from the usual percolation transition, which exhibits linear growth. The mutually connected component never appears with a continuous transition in such networks, for any number of layers. For three or more layers, the giant component appears with a discontinuous hybrid transition, of the same kind found in the mutually connected component of multiplex networks (Baxter et al., 2012) and in k-core percolation (Baxter, Dorogovtsev, Goltsev, & Mendes, 2011; Dorogovtsev et al., 2006). The size of the giant component S jumps from zero to a finite value at the critical point, and there is a square root singularity above the transition.

2.1 Self-Consistency Equations for Weak Multiplex Percolation

To understand the critical behaviour of weak multiplex percolation, let us consider a large sparse random multiplex network, consisting of N nodes connected in M layers (colours), each having its own unique type of edge. This is equivalent to considering a network with M different types of edges connecting the nodes. For clarity, we refer to the different types of edges as different *colours*. Note that a node does not necessarily participate in all layers. Nodes which initially have no connections in a given layer are considered not to participate in that layer. A node is considered active if it maintains at least one connection to another active node in each of the layers in which it participates. A weak percolating cluster is then a set of such active nodes which are connected to each other (each member is connected to at least one other member in at least one layer).

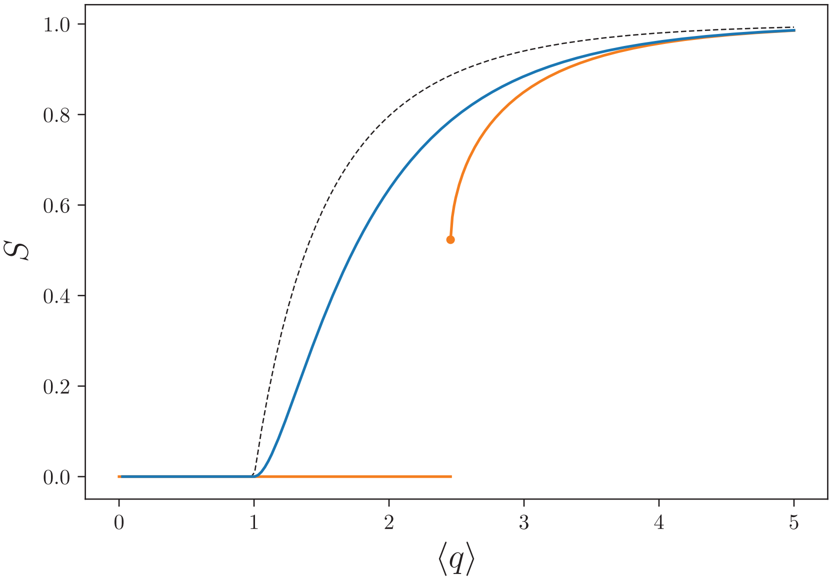

Figure 2 Relative size of the weak multiplex percolation giant component as a function of mean degree in Erdős–Rényi networks, for $M = 2$ layers (blue), showing the continuous phase transition, and $M = 3$ layers (orange) showing the discontinuous hybrid transition. For comparison, the relative size of the single-layer Erdős–Rényi network giant component is also shown (dashed line).

We consider a generalised configuration model, defined by its joint degree distribution $P(q_1, q_2, \ldots, q_M)$. This allows for arbitrary degree correlations between layers, which do not at all impede the analysis. Each layer l is therefore a random graph, defined by its internal degree distribution $P(q_l) \equiv \sum_{i \neq l} \sum_{q_i} P(q_1, q_2, \ldots, q_M)$. This formulation does not, however, consider degree–degree correlations within layers, although one may generalise the analysis to consider them. Note that in the analysis that follows we do not explicitly consider either site or bond percolation. These may easily be considered, however, by taking into account how the joint degree distribution is modified as a function of the control parameter p (representing the fraction of occupied nodes or edges, respectively) and substituting into the relevant equations.

As the number of nodes N tends to infinity, the relative prevalence of finite loops in each layer tends to zero, and each layer can be considered locally treelike. This property allows us to write self-consistency equations to calculate the relative size of the giant weak percolation cluster. The advantage of dealing with a treelike network is that we may consider the connectivity of each neighbour of a given node to be independent of the other neighbours. Furthermore, in a configuration model network with no neighbour degree correlations, an edge emanating from a randomly selected node of degree q leads to a node of degree q' with probability $q'P(q')/\langle q \rangle$.

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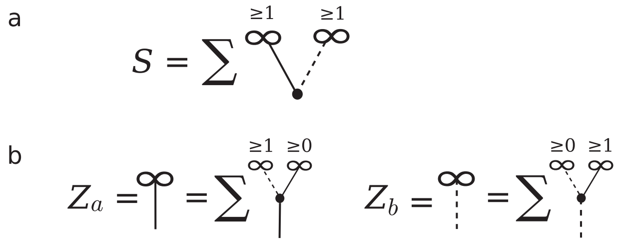

Figure 3 Diagrammatic representation of self-consistency equations in a multiplex network with $M = 2$ treelike layers, labelled a and b. (a) A node belongs to the giant weak percolation cluster (giant component) if it has at least one connection via an edge of type a satisfying the configuration which occurs with probability Z_a (represented by a solid edge leading to an infinity symbol), and one of type b satisfying Z_b (represented by dashed edge leading to an infinity symbol). This corresponds to Eq. (2.1). (b) The probabilities Z_a (left) and Z_b (right) obey recursive relations corresponding to Eq. (2.2). For Z_a, an edge in layer a leads to a node with at least one outgoing edge of type b satisfying Z_b, and similarly for Z_b.

We can then write a self-consistency equation for the probability S that a randomly selected node is active, that is, that it belongs to a weak percolation cluster. This occurs if, in each layer $l = 1, \ldots, M$, the node has at least one neighbour which fulfils the required configuration, which we will define in a moment, and which occurs with probability Z_l in layer l. This condition is represented diagrammatically for two layers in Figure 3(a). Since the probability for each neighbour is independent, the probability that it occurs in at least one neighbour is simply a sum of binomial factors. We can thus write

$$\begin{aligned} S &= \sum_{q_1,q_2,\ldots,q_M} P(q_1, q_2, \ldots, q_M) \prod_{l=1}^{M} \sum_{m=1}^{q_l} \binom{q_l}{m} Z_l^m (1 - Z_l)^{q_l - m} \\ &= \sum_{q_1,q_2,\ldots,q_M} P(q_1, q_2, \ldots, q_M) \prod_{l=1}^{M} [1 - (1 - Z_l)^{q_l}]. \end{aligned} \tag{2.1}$$

To calculate the probabilities Z_l, one may use a recursive argument similar to the one above. Let us imagine following a randomly selected edge in layer l to one of its ends. Clearly, the node we reach has a connection in layer l. It also has connections in each of the other layers with probabilities given by factors of the same form as in the equation for S. Thus, we can write an equation for Z_l purely in terms of the probabilities Z_m:

$$\begin{aligned} Z_l &= \sum_{q_1,q_2,\ldots,q_M} \frac{q_l P(q_1, q_2, \ldots, q_M)}{\langle q_l \rangle} \prod_{m \neq l} [1 - (1 - Z_m)^{q_m}] \\ &\equiv \Psi_l(Z_1, \ldots, Z_{l-1}, Z_{l+1}, \ldots, Z_M), \end{aligned} \tag{2.2}$$

for $l = 1, 2, \ldots, M$. We define the right side of this equation as the function $\Psi_l(Z_1, \ldots, Z_{l-1}, Z_{l+1}, \ldots, Z_M)$. These self-consistency equations are represented diagrammatically, for two layers, in Figure 3(b).

One may also write this equation using a vector notation,

$$\mathbf{Z} = \mathbf{\Psi}(\mathbf{Z})\,, \tag{2.3}$$

where $\mathbf{Z}$ is a vector whose elements are $Z_1, Z_2, \ldots, Z_M$, and $\mathbf{\Psi}$ is a vector function, Ψ_i, $i \in \{1, \ldots, M\}$, of these variables.

As long as the total degree $q_1 + q_2 + \cdots + q_M$ is much smaller than the number of nodes in the network, that is, the projected network is sparse, the relative frequency of finite loops in this projected network is also vanishing. This means that the probability of a node belonging to a finite weak percolation cluster tends to zero. Thus, the probability S is also the probability that a node belongs to a giant weak percolation cluster (which we will refer to henceforth simply as the ‘giant component’). This is the same as the relative size of the giant component. One may thus obtain the size of the giant percolating cluster by first solving simultaneously the recursive Eq. (2.2), then substituting into Eq. (2.1).

Consider for a moment the simplest case of a symmetric uncorrelated multiplex network, in which all layers have the same degree distribution, with no degree correlations between layers. Then we need only a single variable Z, which is the same for all layers. In this case, $M = 2$ layers $\Psi(Z)$ is a concave function of Z, meaning that the weak percolation giant component appears with a continuous transition, while in three or more layers it is a convex function, and the transition becomes discontinuous. We demonstrate these critical phenomena in detail in Section 4.

3 Relation to Other Percolation Models

At this point it is instructive to compare the weak percolation model with two related models: percolation in a single-layer network, and the mutually connected component in multiplex or multilayer networks. Both may be examined through the use of self-consistency equations, in a way very similar to what we have already done for weak multiplex percolation. A brief summary of these two models highlights the differences in the critical behaviour, and hence the unique properties of weak multiplex percolation.

3.1 Percolation in a Single-Layer Network

In the limit of a single layer, $M = 1$, both weak multiplex percolation and the mutually connected component coincide with the usual percolation in a network. Two nodes belong to the same cluster if there is at least one path between them. One may use the mean degree of a random network as a control parameter, or alternatively one may apply random damage to a given network, retaining a fraction p of nodes (site percolation) or of all edges (bond percolation) in the network. A giant connected component appears at a critical value of the control parameter and typically grows linearly with the distance above the critical point, although nonlinear exponents may appear in strongly heterogeneous networks (Dorogovtsev et al., 2008).

In large sparse uncorrelated random networks, one may use the tree ansatz to write self-consistency equations for the relative size of the giant connected component, just as we have done for weak multiplex percolation. A node belongs to the giant component if it has at least one edge leading to an infinite subtree, which occurs with probability X:

$$S = \sum_q P(q)\left[1 - (1 - X)^q\right], \tag{3.1}$$

where X obeys the recursive equation

$$X = \sum_q \frac{qP(q)}{\langle q \rangle}\left[1 - (1 - X)^{q-1}\right]; \tag{3.2}$$

compare Eq. (2.2).

Linearising Eq. (3.2), we find the criterion for the critical threshold,

$$\langle q \rangle = \langle q(q-1) \rangle. \tag{3.3}$$

Expanding Eq. (3.2) for small X and now keeping terms up to second order allows us to find the behaviour near the critical point. We find

$$X = \frac{\langle q(q-1) \rangle - \langle q \rangle}{\langle q(q-1)(q-2) \rangle}, \tag{3.4}$$

and thus, using the expansion of Eq. (3.1),

$$S = \frac{\langle q \rangle[\langle q(q-1) \rangle - \langle q \rangle]}{\langle q(q-1)(q-2) \rangle}. \tag{3.5}$$

Considering, for example, the mean degree $\langle q \rangle$ as a control parameter, and comparing with Eq. (3.3), we see that the giant component grows linearly above the critical point; see Figure 2. This is the classical percolation transition: a continuous second-order transition with linear growth above the critical point.

3.2 The Mutually Connected Component in Multilayer Networks

The mutually connected clusters in a multiplex or multilayer interdependent network are groups of nodes, each pair of which is connected by a path in