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Wigner-Type Theorems for Hilbert Grassmannians

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Preface

Wigner's theorem [67] provides a geometric characterization of unitary and anti-unitary operators as transformations of the set of rays of a complex Hilbert space or, equivalently, rank one projections. This statement plays an important role in the mathematical foundations of quantum mechanics [11, 50, 63], since rays (rank one projections) can be identified with pure states of quantum mechanical systems. We present various types of extensions of Wigner's theorem onto Hilbert Grassmannians and their applications. Most of these results were obtained after 2000, but for completeness of the exposition we include some classic theorems closely connected to the main topic (for example, Kakutani and Mackey's result on the lattice of closed subspaces of a complex Banach space [31] and Kadison's theorem on transformations preserving the convex structure of the set of states of quantum mechanical systems [30]). We use geometric methods related to the Fundamental Theorem of Projective Geometry and results in the spirit of Chow's theorem [13].

Formally, the material is based on the first part of Varadarajan's book [63, Chapters I – IV], but we do not assume that readers are familiar with this book. So, we describe briefly all relevant facts from the mathematical foundations of quantum mechanics and refer to Birkhoff and von Neumann [6] for motivations, physical background and the detailed description. Since the requirement of readers is only knowledge of the basics of linear algebra and operator theory, the book is accessible for graduate students.

I am very grateful to Antonio Pasini for useful discussions on some parts of this book. Finally, I would like to express my sincere gratitude to the anonymous reviewer whose remarks, suggestions and corrections helped me to finish the book.

Mark Pankov

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