Cambridge University Press
978-1-108-79091-8 — Wigner-Type Theorems for Hilbert Grassmannians
Mark Pankov
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Introduction

It was first observed by Birkhoff and von Neumann [6] that the logical structure of quantum mechanics is related to the orthomodular lattice formed by closed subspaces of a complex Hilbert space. On each orthomodular lattice is defined an important class of functions called *states*; all states form a convex set whose extreme points are known as *pure states* [63, Section III.3]. Gleason's theorem [26] describes the set of states for the orthomodular lattices associated to separable complex Hilbert spaces. It says that all states can be identified with bounded self-adjoint positive operators of trace one; in particular, pure states correspond to rank one projections, i.e. rays of the Hilbert space.

The classic Wigner's theorem [67] (see also [11, 50, 63]) characterizes unitary and anti-unitary operators as symmetries of quantum mechanical systems, i.e. every bijective transformation of the set of pure states preserving the transition probability is induced by a unitary or anti-unitary operator. We refer to Chevalier [12] for a history and a brief description of the physical background (for example, it is shown how to derive the Schrödinger equation for a conservative physical system from Wigner's theorem).

In this book, readers will meet two versions of Wigner's theorem. The non-bijective version says that an arbitrary transformation of the Grassmannian formed by rays of a complex Hilbert space which preserves the angles between pairs of rays (the square of the cosine of such an angle is equal to the transition probability between the corresponding pure states) is induced by a linear or conjugate-linear isometry. On the other hand, it was observed by Uhlhorn [62] that to get a unitary or anti-unitary operator it is sufficient to require that a transformation of the Grassmannian of rays is a bijection preserving the orthogonality relation in both directions and the dimension of the Hilbert space is not less than three. Note that the non-bijective analogue of the latter statement does not hold for infinite-dimensional Hilbert spaces. Uhlhorn's theorem is a simple consequence of the Fundamental Theorem of Projective Geometry (for this reason, the dimension of the Hilbert space is assumed to be not less than three); but it reveals the following important relation between the logical structure and the probabilistic structure of quantum mechanical systems: if the logical structure is preserved, then probabilistic structure also is preserved.

The description of bijective transformations preserving the convex structure of the set of all quantum states (the set of all bounded self-adjoint positive operators of trace one) [30] is a classic application of Uhlhorn's version of Wigner's theorem. Since pure states are characterized as extreme points of the convex set of all states, every such transformation induces a bijective transformation of the set of pure states. The latter transformation preserves the orthogonality relation in both directions (this fact is non-trivial) and we come to a unitary or anti-unitary operator.

We present Wigner type theorems for Hilbert Grassmannians. It must be pointed out that we distinguish the Grassmannians whose elements are finite-dimensional subspaces (the dual objects are the Grassmannians consisting of closed subspaces of finite codimensions) and the Grassmannians formed by closed subspaces whose dimension and codimension both are infinite. Results of such a kind were first obtained in [36] and [27, 59], where the non-bijective and Uhlhorn's versions of Wigner's theorem were extended on other Grassmannians. Molnár's theorem [36] states that transformations of Grassmannians (formed by finite-dimensional subspaces) preserving the principal angles between any pair of subspaces are induced by linear and conjugate-linear isometries (except one finite-dimensional case). We generalize this result and show that it is sufficient to assume that only some types of the principal angles are preserved. Another generalization of Molnár's theorem was proved by Gehér [25].

Györy [27] and Šemrl [59] (independently) described bijective transformations of Hilbert Grassmannians preserving the orthogonality relation in both directions. Note that a non-bijective version of this result holds only for finite-dimensional Hilbert spaces. One of the applications of the Györy–Šemrl theorem is the determination of isometries of Hilbert Grassmannians with respect to the gap metric [24].

In the case when the Grassmannian consists of closed subspaces whose dimension and codimension both are infinite, a bijective transformation preserving the orthogonality relation (in both directions) is also inclusions preserving and we show that it can be extended to an automorphism of the lattice of closed subspaces. It is well known that all automorphisms of the lattice of closed subspaces of an infinite-dimensional complex normed space are induced by linear and conjugate-linear homeomorphisms of the normed space to itself (for the finite-dimensional case this fails). This fact was established by

Kakutani and Mackey [31] as a step in the proof of the following remarkable result: every orthomodular lattice formed by all closed subspaces of an infinite-dimensional complex Banach space is the orthomodular lattice associated to a complex Hilbert space.

We also investigate compatibility preserving transformations. The compatibility relation is one of the basic concepts of quantum logic. The orthomodular lattice formed by closed subspaces of a complex Hilbert space is considered as the standard quantum logic. Elements of this lattice are identified with projections, i.e. self-adjoint idempotents in the Banach algebra of bounded operators. Two closed subspaces are compatible if and only if the corresponding projections commute. Two distinct rays are compatible only in the case when they are orthogonal. For this reason, we regard statements which describe compatible preserving transformations as Wigner type theorems.

We will use geometric methods based on properties of Grassmann graphs in the spirit of [15, 45]. So, the Fundamental Theorem of Projective Geometry, Chow's theorem [13], apartments and their orthogonal analogues will be useful tools for our investigations. We include a chapter on geometric transformations of Grassmannians associated to vector spaces of arbitrary (not necessarily finite) dimension. A large portion of the results of this chapter is new and cannot be found in [15, 44, 45].

At the end, we give a few words on applications. It was noted above that Uhlhorn's version of Wigner's theorem was exploited in determining bijective transformations preserving the convex structure of the set of all quantum states. In a similar way, we will use analogues of Wigner's theorem for Hilbert Grassmannians to study linear transformations of the real vector space of self-adjoint finite-rank operators which send projections of fixed rank to projections of the same rank [1, 57, 58] or to projections of other fixed rank [49].

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Two Lattices

We describe briefly some basic properties of the lattice formed by all subspaces of a vector space and the orthomodular lattice consisting of all closed subspaces of a complex Hilbert space. The first lattice is investigated in classic projective geometry [3]. The second is related to the logical structure of quantum mechanical systems (we refer to [19, 63] for the details and strongly recommend the short problem book [14] as a quick introduction to the topic).

1.1 Lattices

Let X be a non-empty set with a certain relation denoted by $\leq$. The pair $(X, \leq)$ is called a *partially ordered set* if for all $x, y, z \in X$ the following three conditions hold:

- $x \leq x$;
- if $x \leq y$ and $y \leq x$, then $x = y$;
- if $x \leq y$ and $y \leq z$, then $x \leq z$.

A partially ordered set $(X, \leq)$ is said to be a *lattice* if it satisfies the following additional conditions:

- for any two elements $x, y \in X$ there is the *least upper bound* $x \vee y$, i.e. an element $z \in X$ such that $x \leq z$, $y \leq z$ and we have $z \leq z'$ for all $z' \in X$ satisfying $x \leq z'$ and $y \leq z'$;
- for any two elements $x, y \in X$ there is the *greatest lower bound* $x \wedge y$, i.e. an element $t \in X$ such that $t \leq x$, $t \leq y$ and we have $t' \leq t$ for all $t' \in X$ satisfying $t' \leq x$ and $t' \leq y$.

A lattice is called *bounded* if it contains the *least* element 0 and the *greatest* element 1 such that $0 \leq x \leq 1$ for every element x. A lattice $(X, \leq)$ is *complete* if for every subset $Y \subset X$ there is the least upper bound $\bigvee_{y\in Y} y$ and the greatest lower bound $\bigwedge_{y\in Y} y$.

An *isomorphism* between partially ordered sets $(X, \leq)$ and $(X', \leq)$ is a bijection $f : X \to X'$ preserving the order $\leq$ in both directions, i.e. for $x, y \in X$ we have

$$x \leq y \iff f(x) \leq f(y).$$

If these partially ordered sets are lattices and $f : X \to X'$ is an isomorphism between them, then

$$f(x \vee y) = f(x) \vee f(y) \text{ and } f(x \wedge y) = f(x) \wedge f(x)$$

for all $x, y \in X$; moreover, if our lattices are complete, then

$$f\left(\bigvee_{y\in Y} y\right) = \bigvee_{y\in Y} f(y) \text{ and } f\left(\bigwedge_{y\in Y} y\right) = \bigwedge_{y\in Y} f(y)$$

for any subset $Y \subset X$. Isomorphisms of bounded lattices transfer the least and greatest elements to the least and greatest elements, respectively.

A bijection $g : X \to X'$ is said to be an *anti-isomorphism* of $(X, \leq)$ to $(X', \leq)$ if it is order reversing in both directions, i.e.

$$x \leq y \iff g(y) \leq g(x)$$

for all $x, y \in X$. If our partially ordered sets are lattices and $g : X \to X'$ is an anti-isomorphism between them, then

$$g(x \vee y) = g(x) \wedge g(y) \text{ and } g(x \wedge y) = g(x) \vee g(y)$$

for all $x, y \in X$; also, we have

$$g\left(\bigvee_{y\in Y} Y\right) = \bigwedge_{y\in Y} g(y) \text{ and } g\left(\bigwedge_{y\in Y} Y\right) = \bigvee_{y\in Y} g(y)$$

for any subset $Y \subset X$ if our lattices are complete. Anti-isomorphisms of bounded lattices transpose the least and greatest elements.

Example 1.1 For every non-empty set X we denote by $\mathcal{L}(X)$ the set of all subsets of X. The partially ordered set $(\mathcal{L}(X), \subset)$ is a bounded lattice. If A and B are subsets of X, then their least upper bound is $A \cup B$ and their greatest lower bound is $A \cap B$. The least element of the lattice $\mathcal{L}(X)$ is the empty set and the greatest element is X. This lattice is complete. The lattices $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ are isomorphic if and only if the sets X and Y are of the same cardinality. In this case, every isomorphism of these lattices is induced by a bijection between X and Y.

A bounded lattice is said to be *complemented* if for every element x there is a *complement* x', i.e. an element x' satisfying

$$x \wedge x' = 0 \ \text{ and } \ x \vee x' = 1.$$

A *Boolean algebra* is a complemented lattice with the following distributive rules:

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z),$$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

Using these rules, we can show that for every element of a Boolean algebra there is the unique complement (see, for example, [63, p. 8]).

Example 1.2 The lattice $\mathcal{L}(X)$ from Example 1.1 is a Boolean algebra.

Let $(X, \leq)$ be a bounded lattice. An *orthocomplementation* is a transformation $x \to x^\perp$ such that for all $x, y \in X$ the following conditions hold:

(1) $x \vee x^\perp = 1$ and $x \wedge x^\perp = 0$,
(2) $x^{\perp\perp} = x$,
(3) if $x \leq y$, then $y^\perp \leq x^\perp$.

The conditions (2) and (3) imply that the orthocomplementation is an anti-automorphism of $(X, \leq)$. Hence $0^\perp = 1$ and $1^\perp = 0$. For elements $x, y \in X$ we write $x \perp y$ and say that these elements are *orthogonal* if $x \leq y^\perp$ (this relation is symmetric, since $x \leq y^\perp$ implies that $y \leq x^\perp$).

A bounded lattice with an orthocomplementation is called *orthomodular* if for any two elements x, y satisfying $x \leq y$ we have

$$x \vee (x^\perp \wedge y) = y.$$

In such a lattice, De Morgan's laws

$$(x \vee y)^\perp = x^\perp \wedge y^\perp \ \text{ and } \ (x \wedge y)^\perp = x^\perp \vee y^\perp$$

hold true [63, Lemma 3.1]. Two elements x, y of an orthomodular lattice are said to be *compatible* if

$$x' = x \wedge (x \wedge y)^\perp \ \text{ and } \ y' = y \wedge (x \wedge y)^\perp \tag{1.1}$$

are orthogonal. For example, x and y are compatible if $x \leq y$ or $x \perp y$.

Example 1.3 Every Boolean algebra is an orthomodular lattice whose orthocomplementation is the complementation. Let x, y be elements of a Boolean algebra and let x', y' be as in (1.1). Using the distributive rules and De Morgan's laws, we establish that $x' = x \wedge y^\perp$ and $y' = y \wedge x^\perp$, which implies that

$x' \wedge y' = 0$, i.e. x' and y' are orthogonal. Therefore, any two elements in a Boolean algebra are compatible.

Remark 1.4 By [63, Lemma 3.7], two elements in an orthomodular lattice $(X, \leq)$ are compatible if and only if there is a subset $X' \subset X$ containing these elements and such that $(X', \leq)$ is a Boolean algebra.

Let $(X, \leq)$ be an orthomodular lattice such that for every countable subset there is a least upper bound. A function $p : X \to [0, 1]$ is called a *state* if it satisfies the following conditions:

- $p(0) = 0$ and $p(1) = 1$,
- for every countable subset $\{x_i\}_{i\in I}$ formed by mutually orthogonal elements we have
$$p\left(\bigvee_{i\in I} x_i\right) = \sum_{i\in I} p(x_i).$$

If I is a countable set, $\{p_i\}_{i\in I}$ are states and $\{t_i\}_{i\in I}$ are non-negative real numbers such that $\sum_{i\in I} t_i = 1$, then the function $p : X \to [0, 1]$ defined as

$$p(x) = \sum_{i\in I} t_i p_i(x) \ \text{ for all } \ x \in X$$

is a state, i.e. the set of all states is convex. Extreme points of this convex set are said to be *pure states*. In other words, a state p is pure if for any states p_1, p_2 and any $t \in (0, 1)$ the equality

$$p = tp_1 + (1 - t)p_2$$

implies that $p = p_1 = p_2$.

Example 1.5 Consider the Boolean algebra $\mathcal{L}(X)$ formed by all subsets of a set X. For $x \in X$ we define $p_x(y) = \delta_x^y$ for every $y \in X$ (δ_x^y is the Kronecker symbol) and extend p_x on $\mathcal{L}(X)$ as follows:

$$p_x(A) = \begin{cases} 1 & \text{if} \quad x \in A\,, \\ 0 & \text{if} \quad x \notin A\,; \end{cases}$$

it is clear that p_x is a state. For every state $p : \mathcal{L}(X) \to [0, 1]$ the set

$$A_p = \{x \in X : p(x) > 0\}$$

is countable (otherwise, there is a natural number $n > 1$ such that the set of all $x \in X$ satisfying $p(x) > 1/n$ is uncountable, which is impossible). If X is countable, then p is completely determined by the values on elements of X, i.e.

$$p = \sum_{x\in A_p} t_x p_x\,,$$

where each t_x is greater than 0 and $\sum_{x \in A_p} t_x = 1$ (since $p(A_p) = 1$). In this case, p is a pure state if and only if $p = p_x$ for a certain $x \in X$. In the general case, the same holds if and only if X is a set of non-measurable cardinality [17, Chapter 6, Theorem 1.4].

1.2 The Lattice of Subspaces of a Vector Space

Let V be a left vector space over a division ring R, i.e. V is an additive abelian group (whose identity element is denoted by 0) and there is a left action of the division ring R on V satisfying the following conditions:

(1) $1x = x$ for all $x \in V$,
(2) $a(x + y) = ax + ay$ for all $a \in R$ and $x, y \in V$,
(3) $(a + b)x = ax + bx$ for all $a, b \in R$ and $x \in V$,
(4) $a(bx) = (ab)x$ for all $a, b \in R$ and $x \in V$

(using (2) and (3) we show that $a0 = 0$ for every $a \in R$ and $0 \in V$ and $0x = 0$ for every $x \in V$ and $0 \in R$). This action can be considered as a right action of the *opposite division ring* R^*. The division rings R and R^* have the same set of elements and the same additive operation. The multiplicative operation $a * b$ on R^* is defined as $b \cdot a$, where $\cdot$ is the multiplicative operation on R (note that R coincides with R^* in the commutative case). For the corresponding right action of R^* on V the condition (4) is rewritten as

$$(xb)a = x(b * a).$$

Every left or right vector space over R is a right or, respectively, left vector space over R^*.

Denote by $\mathcal{L}(V)$ the set of all subspaces of V. The partially ordered set $(\mathcal{L}(V), \subset)$ is a bounded lattice. For any two subspaces X and Y the least upper bound is $X + Y$ and the greatest lower bound is $X \cap Y$. The least element is 0 and the greatest element is V. This lattice is complete.

If $\dim V = 1$, then the lattice consists of the least element and the greatest element only. In the case when $\dim V = 2$, every element of $\mathcal{L}(V)$ distinct from 0 and V is a 1-dimensional subspace and for any proper subspaces $X, Y \subset V$ the inclusion $X \subset Y$ implies that the subspaces are coincident. For this reason, we will always suppose that $\dim V \geq 3$.

Remark 1.6 A complemented lattice is called *modular* if for any element x and elements y, z satisfying $y \leq z$ we have

$$(x \vee y) \wedge z = (x \wedge z) \vee y.$$

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The *rank* of a lattice is the maximal number of non-zero elements in linearly ordered subsets. It is well known that a modular lattice of rank $\geq$ 4 is the lattice formed by all subspaces of a left vector space over a division ring if for every element there is more than one complement (it must be pointed out that the rank is not assumed to be finite, see [3, Chapter VII]), and we need the additional desarguesian axiom to state the same for the case of rank three.

If B is a basis of the vector space V, then the set $\mathcal{A}$ consisting of all subspaces spanned by subsets of B is said to be the *apartment* of $\mathcal{L}(V)$ associated to the basis B. The partially ordered set $(\mathcal{A}, \subset)$ is a complete Boolean algebra isomorphic to the Boolean algebra formed by all subsets of a set whose cardinality is the dimension of V. Two bases define the same apartment if and only if the vectors from one basis are scalar multiples of the vectors from the other.

Proposition 1.7 *For any two elements of $\mathcal{L}(V)$ there is an apartment containing them.*

Proof For any two subspaces X, Y we take a basis of $X \cap Y$ and extend it to bases of X and Y. The union of these bases is an independent subset and we extend it to a basis of V. The associated apartment contains both X and Y. □

Remark 1.8 If V is finite-dimensional, then the lattice $\mathcal{L}(V)$ together with the family of all apartments is a structure closely connected to the Tits building of the general linear group $\mathrm{GL}(V)$ (see [61] for the details).

The *Grassmannians* of the vector space V can be defined as the orbits of the action of the general linear group $\mathrm{GL}(V)$ on the lattice $\mathcal{L}(V)$. If V is finite-dimensional, then $\mathcal{G}_k(V)$ is the Grassmannian formed by all k-dimensional subspaces of V, where $1 \leq k \leq \dim V - 1$. Suppose that $\dim V = \alpha$ is an infinite cardinality. For every cardinality $\beta \leq \alpha$ we denote by $\mathcal{G}_\beta(V)$ the Grassmannian consisting of all subspaces $X \subset V$ such that

$$\dim X = \beta \text{ and } \operatorname{codim} X = \alpha,$$

and we write $\mathcal{G}^\beta(V)$ for the Grassmannian formed by all subspaces $Y \subset V$ satisfying

$$\dim Y = \alpha \text{ and } \operatorname{codim} Y = \beta.$$

Then $\mathcal{G}_\alpha(V) = \mathcal{G}^\alpha(V)$ consists of all subspaces whose dimension and codimension both are α. If β is an infinite cardinality and $\mathcal{G}$ is $\mathcal{G}_\beta(V)$ or $\mathcal{G}^\beta(V)$, then for every $X \in \mathcal{G}$ there are infinitely many elements of $\mathcal{G}$ incident to X; we note that the partially ordered set $(\mathcal{G}, \subset)$ is not a lattice. The intersections of a Grassmannian with apartments of $\mathcal{L}(V)$ will be called *apartments* of this Grassmannian.

The *dual vector space* V^* (formed by all linear functionals on V) is a right vector space over R. We will consider V^* as a left vector space over the opposite division ring R^*.

Let $\{e_i\}_{i\in I}$ be a basis of V. Consider the vectors $\{e_i^*\}_{i\in I}$ in V^* satisfying $e_i^*(e_j) = \delta^i_j$ for any pair $i, j \in I$, where δ^i_j is the Kronecker symbol. These vectors form a linearly independent subset of V^*. If V is finite-dimensional, then this is a basis of V^* and we have $\dim V = \dim V^*$. In the case when V is infinite-dimensional, there are elements of V^* which are non-zero on infinitely many e_i and we get $\dim V < \dim V^*$.

Theorem 1.9 *If* $\dim V = \alpha$ *is infinite, then* $\dim V^* = \beta^\alpha$*, where* β *is the cardinality of the associated division ring*[1].

Proof See [3, Section II.3]. □

For every subset $X \subset V$ we define the *annihilator* X^0 as the set of all $x^* \in V^*$ satisfying $x^*(x) = 0$ for all $x \in X$. It is clear that X^0 is a subspace in V^*. For every subset $Y \subset V^*$ the (left) *annihilator* ${}^0Y$ is the subspace of V formed by all vectors $y \in V$ such that $y^*(y) = 0$ for all $y^* \in Y$.

Remark 1.10 If V is finite-dimensional, then the second dual space V^{**} can be naturally identified with V. Every vector $x \in V$ defines the linear functional $x^* \to x^*(x)$ on V^* and this correspondence is a linear isomorphism of V to V^{**}. Then ${}^0X = X^0$ for every subspace $X \subset V^*$.

For every subspace $X \subset V$ we have ${}^0(X^0) = X$, and

$$(X + Y)^0 = X^0 \cap Y^0, \quad (X \cap Y)^0 = X^0 + Y^0$$

for all subspaces $X, Y \subset V$. Similarly, $({}^0X')^0 = X'$ for every subspace $X' \subset V^*$ and

$${}^0(X' + Y') = {}^0X' \cap {}^0Y', \quad {}^0(X' \cap Y') = {}^0X' + {}^0Y'$$

for all subspaces $X', Y' \subset V^*$.

Denote by $\mathcal{L}_{\text{fin}}(V)$ and $\mathcal{L}^{\text{fin}}(V)$ the sets of all subspaces of finite dimension and finite codimension, respectively. If V is finite-dimensional, then $\mathcal{L}_{\text{fin}}(V)$ and $\mathcal{L}^{\text{fin}}(V)$ both are coincident with $\mathcal{L}(V)$. In the case when V is infinite-dimensional, the partially ordered sets $(\mathcal{L}_{\text{fin}}(V), \subset)$ and $(\mathcal{L}^{\text{fin}}(V), \subset)$ are unbounded lattices. The following facts are well known:

- if $X \in \mathcal{L}^{\text{fin}}(V)$, then $X^0 \in \mathcal{L}_{\text{fin}}(V^*)$ and the dimension of X^0 is equal to the codimension of X;

[1] Recall that β^α is the cardinality of the set formed by all maps from a set of cardinality α to a set of cardinality β.