LONDON MATHEMATICAL SOCIETY LECTURE NOTE SERIES

Managing Editor: Professor Endre Süli, Mathematical Institute, University of Oxford,
Woodstock Road, Oxford OX2 6GG, United Kingdom

The titles below are available from booksellers, or from Cambridge University Press at
www.cambridge.org/mathematics

361 Words: Notes on verbal width in groups, D. SEGAL
362 Differential tensor algebras and their module categories, R. BAUTISTA, L. SALMERÓN & R. ZUAZUA
363 Foundations of computational mathematics, Hong Kong 2008, F. CUCKER, A. PINKUS & M.J. TODD (eds)
364 Partial differential equations and fluid mechanics, J.C. ROBINSON & J.L. RODRIGO (eds)
365 Surveys in combinatorics 2009, S. HUCZYNSKA, J.D. MITCHELL & C.M. RONEY-DOUGAL (eds)
366 Highly oscillatory problems, B. ENGQUIST, A. FOKAS, E. HAIRER & A. ISERLES (eds)
367 Random matrices: High dimensional phenomena, G. BLOWER
368 Geometry of Riemann surfaces, F.P. GARDINER, G. GONZÁLEZ-DIEZ & C. KOUROUNIOTIS (eds)
369 Epidemics and rumours in complex networks, M. DRAIEF & L. MASSOULIÉ
370 Theory of *p*-adic distributions, S. ALBEVERIO, A.YU. KHRENNIKOV & V.M. SHELKOVICH
371 Conformal fractals, F. PRZYTYCKI & M. URBAŃSKI
372 Moonshine: The first quarter century and beyond, J. LEPOWSKY, J. MCKAY & M.P. TUITE (eds)
373 Smoothness, regularity and complete intersection, J. MAJADAS & A. G. RODICIO
374 Geometric analysis of hyperbolic differential equations: An introduction, S. ALINHAC
375 Triangulated categories, T. HOLM, P. JØRGENSEN & R. ROUQUIER (eds)
376 Permutation patterns, S. LINTON, N. RUŠKUC & V. VATTER (eds)
377 An introduction to Galois cohomology and its applications, G. BERHUY
378 Probability and mathematical genetics, N. H. BINGHAM & C. M. GOLDIE (eds)
379 Finite and algorithmic model theory, J. ESPARZA, C. MICHAUX & C. STEINHORN (eds)
380 Real and complex singularities, M. MANOEL, M.C. ROMERO FUSTER & C.T.C WALL (eds)
381 Symmetries and integrability of difference equations, D. LEVI, P. OLVER, Z. THOMOVA & P. WINTERNITZ (eds)
382 Forcing with random variables and proof complexity, J. KRAJÍČEK
383 Motivic integration and its interactions with model theory and non-Archimedean geometry I, R. CLUCKERS, J. NICAISE & J. SEBAG (eds)
384 Motivic integration and its interactions with model theory and non-Archimedean geometry II, R. CLUCKERS, J. NICAISE & J. SEBAG (eds)
385 Entropy of hidden Markov processes and connections to dynamical systems, B. MARCUS, K. PETERSEN & T. WEISSMAN (eds)
386 Independence-friendly logic, A.L. MANN, G. SANDU & M. SEVENSTER
387 Groups St Andrews 2009 in Bath I, C.M. CAMPBELL *et al* (eds)
388 Groups St Andrews 2009 in Bath II, C.M. CAMPBELL *et al* (eds)
389 Random fields on the sphere, D. MARINUCCI & G. PECCATI
390 Localization in periodic potentials, D.E. PELINOVSKY
391 Fusion systems in algebra and topology, M. ASCHBACHER, R. KESSAR & B. OLIVER
392 Surveys in combinatorics 2011, R. CHAPMAN (ed)
393 Non-abelian fundamental groups and Iwasawa theory, J. COATES *et al* (eds)
394 Variational problems in differential geometry, R. BIELAWSKI, K. HOUSTON & M. SPEIGHT (eds)
395 How groups grow, A. MANN
396 Arithmetic differential operators over the *p*-adic integers, C.C. RALPH & S.R. SIMANCA
397 Hyperbolic geometry and applications in quantum chaos and cosmology, J. BOLTE & F. STEINER (eds)
398 Mathematical models in contact mechanics, M. SOFONEA & A. MATEI
399 Circuit double cover of graphs, C.-Q. ZHANG
400 Dense sphere packings: a blueprint for formal proofs, T. HALES
401 A double Hall algebra approach to affine quantum Schur–Weyl theory, B. DENG, J. DU & Q. FU
402 Mathematical aspects of fluid mechanics, J.C. ROBINSON, J.L. RODRIGO & W. SADOWSKI (eds)
403 Foundations of computational mathematics, Budapest 2011, F. CUCKER, T. KRICK, A. PINKUS & A. SZANTO (eds)
404 Operator methods for boundary value problems, S. HASSI, H.S.V. DE SNOO & F.H. SZAFRANIEC (eds)
405 Torsors, étale homotopy and applications to rational points, A.N. SKOROBOGATOV (ed)
406 Appalachian set theory, J. CUMMINGS & E. SCHIMMERLING (eds)
407 The maximal subgroups of the low-dimensional finite classical groups, J.N. BRAY, D.F. HOLT & C.M. RONEY-DOUGAL
408 Complexity science: the Warwick master's course, R. BALL, V. KOLOKOLTSOV & R.S. MACKAY (eds)
409 Surveys in combinatorics 2013, S.R. BLACKBURN, S. GERKE & M. WILDON (eds)
410 Representation theory and harmonic analysis of wreath products of finite groups, T. CECCHERINI-SILBERSTEIN, F. SCARABOTTI & F. TOLLI
411 Moduli spaces, L. BRAMBILA-PAZ, O. GARCÍA-PRADA, P. NEWSTEAD & R.P. THOMAS (eds)
412 Automorphisms and equivalence relations in topological dynamics, D.B. ELLIS & R. ELLIS

413 Optimal transportation, Y. OLLIVIER, H. PAJOT & C. VILLANI (eds)
414 Automorphic forms and Galois representations I, F. DIAMOND, P.L. KASSAEI & M. KIM (eds)
415 Automorphic forms and Galois representations II, F. DIAMOND, P.L. KASSAEI & M. KIM (eds)
416 Reversibility in dynamics and group theory, A.G. O'FARRELL & I. SHORT
417 Recent advances in algebraic geometry, C.D. HACON, M. MUSTAŢĂ & M. POPA (eds)
418 The Bloch–Kato conjecture for the Riemann zeta function, J. COATES, A. RAGHURAM, A. SAIKIA & R. SUJATHA (eds)
419 The Cauchy problem for non-Lipschitz semi-linear parabolic partial differential equations, J.C. MEYER & D.J. NEEDHAM
420 Arithmetic and geometry, L. DIEULEFAIT *et al* (eds)
421 O-minimality and Diophantine geometry, G.O. JONES & A.J. WILKIE (eds)
422 Groups St Andrews 2013, C.M. CAMPBELL *et al* (eds)
423 Inequalities for graph eigenvalues, Z. STANIĆ
424 Surveys in combinatorics 2015, A. CZUMAJ *et al* (eds)
425 Geometry, topology and dynamics in negative curvature, C.S. ARAVINDA, F.T. FARRELL & J.-F. LAFONT (eds)
426 Lectures on the theory of water waves, T. BRIDGES, M. GROVES & D. NICHOLLS (eds)
427 Recent advances in Hodge theory, M. KERR & G. PEARLSTEIN (eds)
428 Geometry in a Fréchet context, C.T.J. DODSON, G. GALANIS & E. VASSILIOU
429 Sheaves and functions modulo *p*, L. TAELMAN
430 Recent progress in the theory of the Euler and Navier–Stokes equations, J.C. ROBINSON, J.L. RODRIGO, W. SADOWSKI & A. VIDAL-LÓPEZ (eds)
431 Harmonic and subharmonic function theory on the real hyperbolic ball, M. STOLL
432 Topics in graph automorphisms and reconstruction (2nd Edition), J. LAURI & R. SCAPELLATO
433 Regular and irregular holonomic D-modules, M. KASHIWARA & P. SCHAPIRA
434 Analytic semigroups and semilinear initial boundary value problems (2nd Edition), K. TAIRA
435 Graded rings and graded Grothendieck groups, R. HAZRAT
436 Groups, graphs and random walks, T. CECCHERINI-SILBERSTEIN, M. SALVATORI & E. SAVA-HUSS (eds)
437 Dynamics and analytic number theory, D. BADZIAHIN, A. GORODNIK & N. PEYERIMHOFF (eds)
438 Random walks and heat kernels on graphs, M.T. BARLOW
439 Evolution equations, K. AMMARI & S. GERBI (eds)
440 Surveys in combinatorics 2017, A. CLAESSON *et al* (eds)
441 Polynomials and the mod 2 Steenrod algebra I, G. WALKER & R.M.W. WOOD
442 Polynomials and the mod 2 Steenrod algebra II, G. WALKER & R.M.W. WOOD
443 Asymptotic analysis in general relativity, T. DAUDÉ D. HÄFNER & J.-P. NICOLAS (eds)
444 Geometric and cohomological group theory, P.H. KROPHOLLER, I.J. LEARY, C. MARTÍNEZ-PÉREZ & B.E.A. NUCINKIS (eds)
445 Introduction to hidden semi-Markov models, J. VAN DER HOEK & R.J. ELLIOTT
446 Advances in two-dimensional homotopy and combinatorial group theory, W. METZLER & S. ROSEBROCK (eds)
447 New directions in locally compact groups, P.-E. CAPRACE & N. MONOD (eds)
448 Synthetic differential topology, M.C. BUNGE, F. GAGO & A.M. SAN LUIS
449 Permutation groups and cartesian decompositions, C.E. PRAEGER & C. SCHNEIDER
450 Partial differential equations arising from physics and geometry, M. BEN AYED *et al* (eds)
451 Topological methods in group theory, N. BROADDUS, M. DAVIS, J.-F. LAFONT & I. ORTIZ (eds)
452 Partial differential equations in fluid mechanics, C.L. FEFFERMAN, J.C. ROBINSON & J.L. RODRIGO (eds)
453 Stochastic stability of differential equations in abstract spaces, K. LIU
454 Beyond hyperbolicity, M. HAGEN, R. WEBB & H. WILTON (eds)
455 Groups St Andrews 2017 in Birmingham, C.M. CAMPBELL *et al* (eds)
456 Surveys in combinatorics 2019, A. LO, R. MYCROFT, G. PERARNAU & A. TREGLOWN (eds)
457 Shimura varieties, T. HAINES & M. HARRIS (eds)
458 Integrable systems and algebraic geometry I, R. DONAGI & T. SHASKA (eds)
459 Integrable systems and algebraic geometry II, R. DONAGI & T. SHASKA (eds)
460 Wigner-type theorems for Hilbert Grassmannians, M. PANKOV
461 Analysis and geometry on graphs and manifolds, M. KELLER, D. LENZ & R.K. WOJCIECHOWSKI
462 Zeta and L-functions of varieties and motives, B. KAHN
463 Differential geometry in the large, O. DEARRICOTT *et al* (eds)
464 Lectures on orthogonal polynomials and special functions, H.S. COHL & M.E.H. ISMAIL (eds)
465 Constrained Willmore surfaces, Á.C. QUINTINO
466 Invariance of modules under automorphisms of their envelopes and covers, A.K. SRIVASTAVA, A. TUGANBAEV & P.A. GUIL ASENSIO
467 The genesis of the Langlands program, J. MUELLER & F. SHAHIDI
468 (Co)end calculus, F. LOREGIAN
469 Computational cryptography, J.W. BOS & M. STAM (eds)
470 Surveys in combinatorics 2021, K.K. DABROWSKI *et al* (eds)

London Mathematical Society Lecture Note Series: 468

(Co)end Calculus

FOSCO LOREGIAN
Tallinn University of Technology

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India

79 Anson Road, #06–04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit
of education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781108746120
DOI: 10.1017/9781108778657

First published 2021

Printed in the United Kingdom by TJ Books Limited, Padstow Cornwall

A catalogue record for this publication is available from the British Library.

ISBN 978-1-108-74612-0 Paperback

A `Εκάτερα
ambrosia pura
vestale di fuoco.

Φος

Cambridge University Press
978-1-108-74612-0 — (Co)end Calculus
Fosco Loregian
Frontmatter
More Information

Cambridge University Press
978-1-108-74612-0 — (Co)end Calculus
Fosco Loregian
Frontmatter
More Information

Contents

Preface

> No podemos esperarnos que ningún aspecto de la realidad cambie si seguimos usando los medios [...] de un lenguaje que lleva el peso de toda la negatividad del pasado. El lenguaje está ahí, pero tenemos que limpiarlo, revisarlo y, sobre todo, debemos desconfiar de él.
>
> J. Cortázar

What is (co)end 'calculus'. Coend calculus determines the behaviour of suitable universal objects associated to functors of two variables $T : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$.

The intuition behind the process of attaching a special invariant to such a functor T can be motivated in many ways.

It is well-known that a measurable scalar function $f : X \to \mathbb{R}$ from a measurable space (X, Ω) can be integrated 'against' a measure μ defined on Ω to yield a real number

$$\int_X f(x) d\mu$$

(for example, when X is a smooth space, the measure can be legitimately thought to depend 'contravariantly' on x, as $d\mu$ is a volume form living in the top-degree exterior algebra of X). In a similar fashion, the evaluation map $V^\vee \otimes V \to k$ for a vector space V is a pairing $\langle \zeta, v \rangle = \zeta(v)$ between a vector v and a co-vector $\zeta : V \to k$, which becomes the sum $\sum_i \zeta_i v_i$ once a basis for V, and its dual basis, is chosen and the vector v has coordinates $(v_1, \dots, v_d)$, whereas ζ has coordinates $(\zeta_1, \dots, \zeta_d)$.

At the cost of pushing this analogy further than permitted, a functor $T : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ can be thought of as a generalised form of evaluation of an object of $\mathcal{C}$ against another; the 'quantity' $T(C, C')$ can then be 'integrated' to yield two distinct objects having dual universal properties:

C1. a *coend*, resulting from the symmetrisation along the diagonal of T, i.e. by modding out the coproduct $\coprod_{C\in\mathcal{C}} T(C, C)$ by the equivalence relation generated by the arrow functions $T(\_, C') : \mathcal{C}^{\mathrm{op}}(X, Y) \to \mathcal{D}(T(X, C'), T(Y, C'))$ and $T(C, \_) : \mathcal{C}(X, Y) \to \mathcal{D}(T(C, X), T(C, Y))$;

C2. an *end*, i.e. an object $\int_C T(C, C)$ arising as an 'object of invariants' or 'fixed points' for the same action of T on arrows; by dualisation, if a coend is a quotient of $\coprod_{C\in\mathcal{C}} T(C, C)$, an end is a subobject of the product $\prod_{C\in\mathcal{C}} T(C, C)$.

This also suggests a fruitful analogy with modules over a ring: if a functor $T : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Set}$ is a 'bimodule', which lets $\mathcal{C}$ act once on the left and once on the right on the sets $T(C, C')$, then the end $\int_C T(C, C)$ is the subspace of invariants for the action of $\mathcal{C}$, whereas the coend $\int^C T(C, C)$ is the space of orbits (or 'coinvariants') of said action.

In fact, a rather common way to employ coends is the following: consider a functor $F : \mathcal{C} \to \mathrm{Set}$ (a 'left module') and a functor $G : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ (a 'right module'), and tensor them together into a functor $(C, C') \mapsto GC \times FC'$; the symmetrisation of $F \times G$ yields a *functor tensor product* of F, G as the set

$$F \boxtimes G := \int^C FC \times GC.$$

Note that in this light, the analogy is meaningful: if $\mathcal{C}$ is a single-object category (so a monoid or a group G), such a pair of modules constitutes a pair (X, Y) of a left and a right G-set, and their functor tensor product can be characterised as the product $X \times_G Y$ obtained as the quotient of $X \times G$ for the equivalence relation $(g.x, y) \sim (x, g.y)$, so that $X \times_G Y$ is the universal G-*bilinear* product of sets, in that the 'scalar' $g \in G$ can pass left-to-right from $(g.x, y)$ to $(x, g.y)$ in the quotient. Of course, the terminology works better when X, Y are vector spaces carrying a *linear* representation of G.

Theorems involving ends and coends $\int^C T(C, C)$ and $\int_C T(C, C)$ can now be proved by means of the universal properties that define them; it is easily seen that given $T : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ there exists a category $\bar{\mathcal{C}}$ and a functor $\bar{T} : \bar{\mathcal{C}} \to \mathcal{D}$ such that $\int^C T(C, C) \cong \mathrm{colim}_{\bar{\mathcal{C}}} \bar{T}$ and $\int_C T(C, C) \cong \lim_{\bar{\mathcal{C}}} \bar{T}$. In the example above, the tensor product of a left G-module and a right G-module (here 'module' means 'k-vector space') can be characterised as the coequaliser

$$\bigoplus_{g\in G} X \otimes_k Y \overset{\alpha}{\underset{\beta}{\rightrightarrows}} X \otimes_k Y \longrightarrow X \otimes_G Y \tag{0.1}$$

where $\alpha(g,(x,y)) = (g.x, y)$ and $\beta(g,(x,y)) = (x, g.y)$.

So, all (co)ends can be characterised as (co)limits; but they provide a richer set of computational rules than mere (co)limits. Often, establishing that an object has a certain universal property is a difficult task, because a direct argument tangles the reader into using elements. A general tenet of modern category theory is that cleaner, more conceptual arguments are preferred to element-wise proofs that are evil in spirit, if not in shape.

(Co)end calculus provides such a conceptualisation for many classical arguments of category theory: it is in fact possible to prove that two objects of a category, at least one of which is defined as a coend, are isomorphic by means of a chain of ‘deduction rules’.

These rules are described in the first half of the book, but here we glimpse at what they look like.[1]

In order to make clear what this paragraph is about, let us consider the statement that *right adjoints preserve limits*; it is certainly possible to prove this by hand. Nevertheless, using little more than the Yoneda lemma it is possible to prove that if $R : \mathcal{D} \to \mathcal{C}$ is right adjoint to $L : \mathcal{C} \to \mathcal{D}$, there is a natural isomorphism of hom-sets $\mathcal{C}(C, R(\lim_{\mathcal{J}} D_J)) \cong \mathcal{C}(C, \lim_{\mathcal{J}} RD_J)$ for every object C, and every diagram $D : \mathcal{J} \to \mathcal{D}$ by arguing as follows:

$$\frac{\mathcal{C}(C, R(\lim_{\mathcal{J}} D_J))}{\frac{\mathcal{C}(LC, \lim_{\mathcal{J}} D_J)}{\frac{\lim_{\mathcal{J}} \mathcal{C}(LC, D_J)}{\frac{\lim_{\mathcal{J}} \mathcal{C}(C, RD_J)}{\mathcal{C}(C, \lim_{\mathcal{J}} RD_J)}}}}$$

[1] The somewhat far-fetched conjecture that permeates all the book is that coend calculus provides a higher dimensional version of a deductive system, suited for category theory (see [CW01] for some preliminary steps in this direction), having deduction rules similar to those of Gentzen’s sequent calculus. We will never attempt to turn this enticing conjecture into a theorem, or even to make a precise claim; the interested reader is thus warned that their curiosity will not get satisfaction – not in the present book, at least. We record that the idea that coends categorify logical calculus comes from William Lawvere, and it was first proposed in [Law73].

where each step of this 'deduction' is motivated either by the fact that $L \dashv R$ are adjoint functors, or by the fact that all functors $\mathcal{C}(X, \_)$ preserve limits. Once this is proved, the Yoneda lemma (see A.5.3 for the statement) entails that there is an isomorphism $R(\lim D_J) \cong \lim RD_J$.

A similar argument is a standard way to prove that a certain object, defined (say) as the left adjoint to a certain functor, must admit an 'integral expansion' to which it is canonically isomorphic. For example, in the proof of what we call the *ninja Yoneda lemma* in 2.2.1, we carry out the following computation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathrm{Set}\Big(\int^{C\in\mathcal{C}} KC\times\mathcal{C}(X,C), Y\Big)}{\int_{C\in\mathcal{C}}\mathrm{Set}\big(KC\times\mathcal{C}(X,C),Y\big)}}{\int_{C\in\mathcal{C}}\mathrm{Set}(\mathcal{C}(X,C),\mathrm{Set}(KC,Y))}}{[\mathcal{C},\mathrm{Set}]\big(\mathcal{C}(X,\_),\mathrm{Set}(K\_,Y)\big)}}{\mathrm{Set}(KX,Y)}$$

where each step has to be interpreted as an application of a certain deduction rule that interchanges coends with ends, places them in and out of a hom functor, etc.

The reduction of proofs to a series of deduction steps embodies some sort of 'logical calculus', whose introduction rules resemble formulae such as

$$\mathrm{Cat}(\mathcal{C},\mathcal{D})(F,G) \rightsquigarrow \int_C \mathcal{D}(FC,GC)$$

where the object $\mathrm{Cat}(\mathcal{C},\mathcal{D})(F,G)$ is decomposed into an integral like $\int_C \mathcal{D}(FC,GC)$, and elimination rules look like

$$\int^C FC\times\mathcal{C}(C,X) \rightsquigarrow FX$$

where an integral is packaged into the object FX (of course, the symmetric nature of the canonical isomorphism relation makes all elimination rules reversible into introductions, and vice versa). Altogether, this allows us to derive the validity of a canonical isomorphism as a result of a chain of deductions, in a 'categorified' fashion.

The reader should not, of course, concentrate now on the meaning of these derivations at all; all notation will be duly introduced at the right time; when the statement of our Proposition 2.2.1 is introduced, the chain of deductions above will look almost tautological, and rightly so.

It is clear that, done in this way, category theory acquires an alluring algorithmic nature, and becomes (if not easy, at least) *easier to understand*.

Thus, the 'calculus' arising from these theorems encodes many, if not all, elementary constructions in category theory (we shall see that it subsumes the theory of (co)limits, it allows for a reformulation of the Yoneda lemma, it provides an explicit formula to compute pointwise Kan extensions, and it is a cornerstone of the 'calculus of bimodules', encoding the compositional nature of *profunctors*, the natural categorification of relational composition).

As it stands, (co)end calculus describes pieces of abstract and universal algebra [Cur12, GJ17], algebraic topology [Get09, May72, MSS02], representation theory [LV12], logic, computer science [Kme18], as well as pure category theory. The present book wishes to explore in detail a theory of (co)end calculus and its applications in detail.

So far, we have the motivations for the *topic* of this book. What about the motivation *for the book itself*? It shall be noted that (co)ends are not absent from the already existing literature on category theory: the topic is covered in [ML98], a statutory reading for every categorephile, and Mac Lane himself used coends to characterise a construction in algebraic topology as a 'tensor product' operation between functors in his [ML70]. Coends are mentioned (but not used as widely as they deserve) in Borceux's *Handbook* (in its first two tomes [Bor94a, Bor94b]). However, the topic lacks a treatment that is at the same time systematic, easy to read, and monographic.

As a result, (co)end calculus still lies just beyond the grasp of many people, and even of a few category theorists, because the literature that could teach its simple rules is a vast constellation of scattered papers, drawing from a large number of diverse disciplines.

This situation is all the more an issue because nowadays category theory has fruitfully contaminated with applied sciences. In the opinion of the author, it is of the utmost importance to provide his growing community with a single reference that accounts for the simplicity and unitary nature of category theory through (co)end calculus, thereby providing proof for its plethora of applications, and popularising this 'secret weapon' of category theorists, making it available to novices and non-mathematicians. The present endeavour is but a humble attempt to address this issue.

A brief history of (co)ends. Like many other pieces of mathematics, (co)end calculus was developed as a tool for homological algebra: the first definition of a universal object called '(co)end' was given in a paper studying the Ext functors, and the father of (co)end calculus is none other than Nobuo Yoneda. In [Yon60] he singled out most of the definitions we will introduce in the first five chapters of this book.

Having read Yoneda's original paper in order to write the present introduction, we find no better way than to quote the original text, untouched, just occasionally adding a few details here and there to frame Yoneda's words in a modern perspective, (but also in order to adapt them to our choice of notation).

There are multiple reasons for this strategy: [Yon60] is a mathematical gem, an enticing prelude of all the theory developed in the subsequent decades, and a perfect prelude to the story this book tries to tell. Even more so, in reporting Yoneda's words we believe we are also doing a service to the mathematical community, since the integral text of [Yon60] is somewhat difficult to find.

Our sincere hope is that this introduction, together with the whole book the reader is about to read, credits the visionary genius of Yoneda: category theory has few theorems, and one of them is a lemma. The *Yoneda lemma*, in its myriad incarnations, is certainly a cornerstone of structural thinking, way beyond category theory. If anything more was needed to revere Nobuo Yoneda, let this be (co)end calculus.

The paper [Yon60] starts introducing (co)ends in the following way:

> Let $\mathcal{C}$ be a category. By a *left* $\mathcal{C}$*-group* we mean a covariant functor M of $\mathcal{C}$ with values in the category Ab of abelian groups and homomorphisms. [...] Also by a $\mathcal{C}^*$-group (or a *right* $\mathcal{C}$*-group*) we mean a contravariant functor $K : \mathcal{C} \to \text{Ab}$. [...] Functors of several variables with values in Ab will accordingly be called $\mathcal{B}$-$\mathcal{C}$-groups, $\mathcal{B}^*$-$\mathcal{C}$-groups, etc.
>
> Let H be a $\mathcal{C}^*$-$\mathcal{C}$-group, and G an additive group. By a *balanced homomorphism* $\mu : G \Rightarrow H$ we mean a system of homomorphisms $\mu(C) : G \to H(C, C)$ defined for all objects $C \in \mathcal{C}$ such that for every

map $\gamma : C \to C'$ in $\mathcal{C}$ commutativity holds in the diagram

$$\begin{array}{ccc} G & \xrightarrow{\mu(C)} & H(C,C) \\ {\scriptstyle \mu(C')}\downarrow & & \downarrow{\scriptstyle H(C,\gamma)} \\ H(C',C') & \xrightarrow[H(\gamma,C')]{} & H(C,C'). \end{array}$$

Also by a balanced homomorphism $\lambda : H \Rrightarrow G$ we mean a system of homomorphisms $\lambda(C) : H(C,C) \to G$ defined for all objects $C \in \mathcal{C}$ such that for every map $\gamma : C \to C'$ in $\mathcal{C}$ commutativity holds in the diagram

$$\begin{array}{ccc} H(C',C) & \xrightarrow{H(C',\gamma)} & H(C',C') \\ {\scriptstyle H(\gamma,C)}\downarrow & & \downarrow{\scriptstyle \lambda(C')} \\ H(C,C) & \xrightarrow[\lambda(C)]{} & G \end{array}$$

Of course, here a left/right '$\mathcal{C}$-group' is merely an Ab-enriched presheaf (covariant or contravariant) with domain $\mathcal{C}$. The above paragraphs define the fundamental notions we will use throughout the entire book: *wedges* and *cowedges*. These are exactly natural maps to/from a constant, which vary taking into account the fact that $H(C,C)$ depends both covariantly and contravariantly on C. There is a category of such (co)wedges, and a process dubbed *(co)integration* picks the initial and terminal objects of such categories:

> An additive group I together with a balanced $\theta : H \Rrightarrow I$ is called *integration* of a $\mathcal{C}^*$-$\mathcal{C}$-group H if it is universal among balanced homomorphisms from H, i.e. if for any other balanced homomorphism $\lambda : H \Rrightarrow G$ there is a unique morphism $\zeta : I \to G$ such that $\zeta \circ \theta(C) = \lambda(C)$ for every object $C \in \mathcal{C}$.
>
> [Integrations and cointegrations] are [...] given as follows: for a map $\gamma : C \to C'$ in $\mathcal{C}$ we put $H(\gamma) = H(C',C)$, $H(\gamma^*) = H(C,C')$, and define homomorphisms
>
> $$\partial_\gamma : H(\gamma) \to H(C,C) \oplus H(C',C')$$
> $$\delta_\gamma : H(C,C) \oplus H(C',C') \to H(\gamma^*)$$
>
> by
>
> $$\partial_\gamma(h') = h' \circ \gamma \oplus (-\gamma \circ h')$$
> $$\delta_\gamma(h \oplus h'') = \gamma \circ h - h'' \circ \gamma$$
>
> Denote by Σ_0 and Π^0 the direct sum $\sum_{C \in \mathcal{C}} H(C,C)$ and the direct product $\prod_{C \in \mathcal{C}} H(C,C)$ respectively. Also, denote by Σ_1 and Π^1 the

direct sum $\sum_{\gamma\in\hom(\mathcal{C})} H(\gamma)$ and $\prod_{\gamma\in\hom(\mathcal{C})} H(\gamma^*)$ respectively. Then $\partial_\gamma, \delta_\gamma$ are extended to homomorphisms

$$\partial : \Sigma_1 \to \Sigma_0 \qquad \delta_{CC'} : \Pi^0 \to \Pi^1.$$

Now [Yon60] proves that the 'integration' $\int_C H$ and the 'cointegration' $\int_C^* H$ of a $\mathcal{C}^*$-$\mathcal{C}$-group are given respectively by the cokernel of $\partial_{CC'}$, and by the kernel of $\delta_{CC'}$, for suitably defined maps $\partial_{CC'}$ and $\delta_{CC'}$.

To avoid confusion, we stress that in this definition Yoneda employed the *opposite* choice of terminology that we will introduce later on: an *integration* for H is a coend $\int^C H(C,C)$, but Yoneda denotes it $\int_C H(C,C)$; a *cointegration* is an end $\int_C H(C,C)$, but Yoneda denotes it $\int_C^* H(C,C)$. Always be careful if you consult [Yon60] or references from the same era.

The coend $\int^C H$ of a functor $H : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{D}$ is a colimit (in the particular case of Ab-enriched functors, a cokernel) built out of H, and the end $\int_C H$ is a limit; precisely, the kernel of a certain group homomorphism. Once this terminology has been set up, given two functors $F, G : \mathcal{C} \to \mathcal{D}$, if we let H be the functor $(C, C') \mapsto \mathcal{D}(FC, GC')$, then a family of arrows $\alpha_C : FC \to GC$ forms the components of a natural transformation $\alpha : F \Rightarrow G$ if and only if the components $\alpha_C : FC \to GC$ lie in the kernel of a 'differential' $\bar{\delta} : \prod_{C\in\mathcal{C}} H(C,C) \to \prod_{\gamma\in\hom(\mathcal{C})} H(\gamma^*)$, obtained in the obvious way 'gluing' all the $\delta_{CC'}$ together.

In dealing with functors of more variables, we shall often inscribe x (or y, z) to indicate the two entries to be considered in the (co)integration, namely we write

$$\int_{X\in\mathcal{C}} H(\dots, X, \dots, X, \dots). \tag{0.2}$$

This is based on the following fact: let H, H' be $\mathcal{C}$-$\mathcal{C}^*$-groups, and let $\theta : H \to \int_{\mathcal{C}} H$, $\theta' : H' \to \int_{\mathcal{C}} H'$ be the integrations. Then a natural transformation $\eta : H \Rightarrow H'$ induces a unique homomorphism $\int_{\mathcal{C}} \eta : \int_{\mathcal{C}} H \to \int_{\mathcal{C}} H'$ such that $\left(\int_{\mathcal{C}} \eta\right) \circ \theta(C) = \theta'(C) \circ \eta(C,C)$. Thus if H is a $\mathcal{B}$-$\mathcal{C}^*$-$\mathcal{C}$-group, then $\int_{X\in\mathcal{C}}^{(*)} H(B,X,X)$ is a $\mathcal{B}$-group. On this account, for an $\mathcal{A}$-$\mathcal{B}^*$-$\mathcal{B}$-$\mathcal{C}$-$\mathcal{C}^*$-group H we have

$$\int_{Y\in\mathcal{B}} \int_{X\in\mathcal{C}} H(A,Y,Y,X,X) = \int_{X\in\mathcal{C}} \int_{Y\in\mathcal{B}} H(A,Y,Y,X,X).$$

Here Yoneda introduces one of the pillars of coend calculus, the *Fubini rule*, i.e. the fact that the result of a (co)integration is the same regardless of the order of integration; this is ultimately just a consequence of the functoriality of the assignment sending a $\mathcal{C}$-$\mathcal{C}^*$-group H into its (co)integration. Of course, there is nothing special about the codomain of H being the category of abelian groups: any sufficiently (co)complete category $\mathcal{D}$ will do, as long as the (co)integrations involved exist.

In modern terms, the Fubini rule can be obtained as a consequence of a much deeper, and hopefully more enlightening, result: we prove it in our Theorem 1.3.1.

> Next for a $\mathcal{B}$-group M and a $\mathcal{C}$-group N, $M \otimes N : N(C) \otimes M(B)$ is a $\mathcal{B}$-$\mathcal{C}$-group, and $\underline{\hom}(M, N) = \hom(MB, NC)$ is a $\mathcal{B}^*$-$\mathcal{C}$-group. For an $\mathcal{A}$-group M and a $\mathcal{B}$-$\mathcal{C}$-$\mathcal{C}^*$-group H we have:
>
> $$\int_{X \in \mathcal{C}} MA \otimes H(B, X, X) = MA \otimes \int_{X \in \mathcal{C}} H(B, X, X)$$
> $$\int_{X \in \mathcal{C}}^{*} \hom(MA, H(B, X, X)) = \hom\left(MA, \int_{X \in \mathcal{C}}^{*} H(B, X, X)\right)$$
> $$\int_{X \in \mathcal{C}}^{*} \hom(H(B, X, X), MA) = \hom\left(\int_{X \in \mathcal{C}}^{*} H(B, X, X), MA\right)$$

As an immediate consequence of these statements we get another fundamental building-block of a coend 'calculus': given two functors $M : \mathcal{B} \to \mathrm{Ab}$ and $N : \mathcal{B}^{\mathrm{op}} \to \mathrm{Ab}$, they can be *tensored* by the integration $M \boxtimes N := \int_{B \in \mathcal{B}} N(B) \otimes_{\mathbb{Z}} M(B)$.

A rather interesting perspective on this construction is the following: the result remains true when the Ab-category $\mathcal{B}$ has a single object, so it is merely a ring B. In such a case, a functor $M : B \to \mathrm{Ab}$ is a left module, and a functor $N : B^{\mathrm{op}} \to \mathrm{Ab}$ is a right module. The integration (or in modern terms, the coend) $\int^{B} N \otimes_{\mathbb{Z}} M$ in this case is exactly the tensor product of B-modules: it has the universal property of the cokernel of the map

$$\bigoplus_{b \in B} M \otimes_{\mathbb{Z}} N \xrightarrow{\varrho} M \otimes_{\mathbb{Z}} N$$

defined by $\varrho(b, m, n) = b.m - n.b$.

As the reader might now suspect, few analogies are more fruitful than the one between modules over which a monoid object acts, and presheaves $\mathcal{C} \to \mathrm{Set}$.

Structure of the book. We shall now briefly review the structure of the book. In the first three chapters we outline the basic rules of (co)end calculus. After having defined (co)ends as universal objects and having proved that they can be characterised as (co)limits, we start denoting such objects as integrals $\int_C T(C, C)$ or $\int^C T(C, C)$. This notation is motivated by the fact that (co)ends 'behave like integrals' in that a *Fubini rule* of exchange holds, see Theorem 1.3.1.

Then in Chapter 2 we introduce the first rules of the calculus: the Yoneda lemma A.5.3 can be restated in terms of a certain coend computation, and pointwise Kan extensions can be computed by means of a (co)end.

After this, we study the particular case of left Kan extensions along the Yoneda embedding: in some sense, the theory of such extensions alone embodies category theory.

The subsequent chapters begin to introduce more modern topics described by means of (co)end calculus. The theories of *weighted (co)limits* (Chapter 4), of *profunctors* (Chapter 5) and *operads* (Chapter 6) are cornerstones of 'formal' approaches to category theory. Weighted (co)limits are the correct notion of (co)limit in an enriched or formal-categorical (see 2.4 and [Gra80]) setting; profunctors are a bicategory where one can re-enact all of category theory, and are deeply linked to categorical algebra and representation theory; operads, initially introduced as a technical means to solve an open problem in homotopy theory, now constitute the common ground where universal algebra and algebraic topology meet. The final point will be Theorem 6.4.7, where we draw a tight link between profunctors and operads.

Chapter 7 studies higher dimensional analogues of (co)ends; first, we study (co)ends in 2-categories; then we move up to infinity and study homotopy-coherent analogues of (co)ends in simplicial categories [Ber07], quasicategories [Lur09], model categories [Hov99] and derivators [Gro13].

Appendix A serves as a short introduction to category theory: it fixes the notation we employ in the previous chapters. A basic knowledge of elementary mathematics is a prerequisite, but we will introduce most categorical jargon from scratch.

Each chapter has a short introduction, in the form of a small abstract; this allows the interested reader to get a glimpse into the content and fundamental results of each chapter (often, one or two main theorems). We believe this format is easier to consult than a comprehensive survey of each chapter given all at once in the introduction, so we felt free to keep this introductory account of the content of the book pretty brief.

Several exercises follow each chapter of the book; there are questions of every level, sometimes easy, sometimes more difficult. Some of them make the reader rapidly acquainted with the computational approach to category theory offered by (co)end calculus; some others shed a new light

Cambridge University Press
978-1-108-74612-0 — (Co)end Calculus
Fosco Loregian
Frontmatter
More Information

on old notions. In approaching them, we advise you to avoid element-wise reasoning; instead, find either an abstract argument, or a 'deduction-style' one.

Some of the exercises are marked with a ◉◉ symbol (eyes wide with fear): this means they are more difficult and less well posed questions than the others. This might be deliberate (and thus part of the exercise is understanding what the question is) or not (and thus the question and its answer are not completely clear even to the author). In the second case, it is likely that a complete answer might result in new mathematics that the solvers are encouraged to develop.

Other kinds of 'small eyes' are present throughout the book: the paragraphs decorated with a ◓◓ contain material that can be skipped at first reading, or material that deepens a prior topic in a not-so-interesting detour; ◐◐ is used to signal key remarks and more generally important material that we ask the reader to digest properly and analyse in full detail.

Notation. Having to deal with many different sources for this exposition, to hope to maintain a coherent choice of notation throughout the whole book is wishful thinking; however, the author did his best to provide a coherent enough one, striving to make it at the same time expressive and simple.

In general, 1-dimensional category-like structures will be denoted as calligraphic letters $\mathcal{C}, \mathcal{D}, \dots$; objects of $\mathcal{C}$ are denoted $C, C', \dots \in \mathcal{C}$. In contrast, 2-categories are often denoted with a sans-serif case $\mathsf{Cat}, \mathsf{K}, \mathsf{A}, \dots$; an object of the 2-category of small categories is denoted $\mathcal{C} \in \mathsf{Cat}$, but an object of an abstract 2-category is denoted $A \in \mathsf{K}$.

Functors between categories are denoted as capital Latin letters such as F, G, H, K (although there can be small deviations from this rule); the category of functors $\mathcal{C} \to \mathcal{D}$ between two categories is almost always denoted as $\mathrm{Cat}(\mathcal{C}, \mathcal{D})$ (or less often $[\mathcal{C}, \mathcal{D}]$; this will be done especially when $[\mathcal{C}, \mathcal{D}]$ is regarded as the internal hom of the closed structure in Cat, or when it is necessary to save some space); the symbols $\_$, $=$ are used as placeholders for the 'generic argument' of a functor or bifunctor (they mark temporal precedence of saturation of a variable); morphisms in the category $\mathrm{Cat}(\mathcal{C}, \mathcal{D})$ (i.e. natural transformations between functors) are often written in lowercase Greek, or lowercase Latin alphabet, and collected in the set $\mathrm{Cat}(\mathcal{C}, \mathcal{D})(F, G)$.

The simplex category $\boldsymbol{\Delta}$ is the *topologist's delta* (as opposed to the *algebraist's delta* $\boldsymbol{\Delta}_+$ which has an additional initial object $[-1] := \varnothing$), having objects *non-empty* finite ordinals $[n] := \{0 < 1 \cdots < n\}$; we denote $\Delta[n]$ the representable presheaf on $[n] \in \mathbf{\Delta}$, i.e. the image of $[n]$ under the Yoneda embedding of $\mathbf{\Delta}$ in the category $\mathsf{sSet} = \widehat{\mathbf{\Delta}}$ of simplicial sets. More generally, we indicate the Yoneda embedding of a category $\mathcal{C}$ into its presheaf category with よ$_{\mathcal{C}}$, or simply よ, i.e. with the hiragana symbol for 'yo'.

Acknowledgments. Writing this book spanned the last five years of my life; it would probably take five more years to do it in the form the topic deserves. I did my best, and if I managed to be proud of the result, it is because I have been helped a lot.

The draft of this book accompanied me to all sorts of places: this has made the list of people who helped and guided me through it quite long. After all this time, it remains all the more true today that 'in some sense, I am not the only author of this note', as I used to write in the early stages of this project.

Several people participated and extensively commented on such drafts; each of them helped to make the present book a little better than it would have been if I had been alone.

I would like to thank S. Ariotta (for providing more help than I deserved, in the proof of the ∞-Fubini rule in 7.3), J. D. Christensen, I. Di Liberti, T. de Oliveira Santos (for an attentive proofreading when my eyes were bleeding), D. Fiorenza (a friend and an advisor I will never be able to refund for his constant support), A. Gagna, N. Gambino (for offering me the opportunity to discuss the content of this book in front of his students in Leeds; in just a few days I realised years of meditation were still insufficient to teach (co)end calculus), L. (for accompanying me there, and for being the best *mate* for a much longer trip), F. Genovese (for believing in (co)end calculus with all his heart, and for making dg-categories more understandable through it), A. Joyal (for letting me sketch the statement of 4.2.2 on the back of a napkin in a cafe in Paris), G. Marchetti, A. Mazel-Gee, B. Milewski, G. Mossa, D. Palombi (for sharing with me his truffles, his friendship, his monoidal categories, and his typographical kinks), D. Ravenel (for believing in this book's draft title, 'This is the (co)end, my only (co)friend' more than I will ever do), M. Román, G. Ronchi (for accepting my harsh ways while I was learning how to be a mentor, and because he Believes that mathematics is Beautiful), J. Rosický (for he trusted in me and gave me pearls of wisdom), E. Riehl, E. Rivas, T. Trimble, M. Vergura, C. Williams, K. Wright, N. Yanofsky,

and countless other people who pointed out mistakes, faulty arguments, typos.

A separate thanks goes to S., P., G., C.: they opened their doors to me when I was frail and broken-hearted. Writing means, always, opening a wound, exposing your tendons to the blade of a sharp knife. During these years I have been protected by the warmth of your smile, of your houses, of your brilliance, of your friendship.

Various institutions supported me through the many years necessary to finish this book. I would like to thank the University of Western Ontario, Masaryk University, The Max Planck Institute for mathematics in Bonn for providing me time and money to keep writing, and the University of Coimbra, which added to the deal a *completely* distraction-free environment. Additional help came from Taltech, where I am working while this book approaches its final form, and from all Estonia: *aitäh!*

Ringrazio per ultimo mio padre, che non comprenderebbe questa dedica se non la scrivessi in italiano, e che non leggerà mai questo libro 'assorto com'è in altra e lontana scienza'. A sei anni mi fece sedere sulle sue ginocchia, e scrisse su un foglio il numero 1 seguito da nove zeri, per farmi capire quanto è grande un miliardo: si può dire tutto questo sia iniziato lì.

Grazie a tutti.