1
Dinaturality and (Co)ends

SUMMARY. Naturality of a family of morphisms $\alpha_C : FC \to GC$ defines the correct notion of map between functors F, G; yet it is not able to describe more subtle interactions that can occur between F and G, for example when both functors have a product category like $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ as domain. A transformation that takes into account the fact that F, G act on morphisms once covariantly and once contravariantly is called *dinatural*.

As ill-behaved as it may seem (in general, dinatural transformations cannot be composed), this notion leads to the definition of a *(co)wedge* and *(co)end* for a functor $T : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$: a dinatural transformation having constant (co)domain, and a suitable universal property. This is in perfect analogy with the theory of (co)limits: universal natural transformations from/to a constant functor. Unlike colimits, however, (co)ends support a *calculus*, that is, a set of inference rules allowing mechanical proof of non-trivial statements as initial and terminal points of a chain of deductions.

The purpose of this chapter, and indeed of the entire book, is to familiarise its readers with the rules of calculus.

> Los idealistas arguyen que las salas hexagonales son una forma necesaria del espacio absoluto o, por lo menos, de nuestra intuición del espacio.
>
> J.L. Borges *La biblioteca de Babel*

1.1 Supernaturality

We choose to let the name 'supernaturality' describe the two sorts of generalisations of naturality for functors that we will investigate throughout the book: *di*naturality, in 1.1.1, and *extra*naturality, in 1.1.8.

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1.1.1 Dinaturality

This first section starts with a simple example. We denote by Set the category of sets and functions, considered with its natural cartesian closed structure (see A.4.3.AD4). This means that we have a bijection of sets

$$\mathrm{Set}(A \times B, C) \cong \mathrm{Set}(A, C^B) \tag{1.1}$$

that is natural in all three arguments, where we let C^B denote the set of functions $f : B \to C$. The bijection above is defined by the maps

$$(f : A \times B \to C) \mapsto A \xrightarrow{\eta_{A,(B)}} (A \times B)^B \xrightarrow{f^B} C^B$$
$$(g : A \to C^B) \mapsto A \times B \xrightarrow{g \times B} C^B \times B \xrightarrow{\epsilon_{C,(B)}} C$$

by means of suitable *unit* and *counit* maps η and ϵ (see A.4.1) witnessing the adjunction. Let us concentrate on the counit map alone (a dual reasoning will yield similar conclusions for the unit): it is a natural transformations having components

$$\{\epsilon_{X,(B)} : X^B \times B \to X \mid X \in \mathrm{Set}\}. \tag{1.2}$$

This family of functions sends a pair $(f, b) \in X^B \times B$ to the element $fb \in X$, and thus deserves the name of *evaluation*.

For the purpose of our discussion, we shall consider this family of morphisms not only natural in X (as every counit morphism), but also mutely depending on the variable B in its codomain. This means that $X^B \times B$ is the image of the pair (B, B) under the functor $(U, V) \mapsto X^U \times V$, and X can be regarded similarly as the image of (B, B) under the constant functor in X. Both functors thus have 'type' $\mathrm{Set}^{\mathrm{op}} \times \mathrm{Set} \to \mathrm{Set}$.

The evaluation maps $\epsilon_{X,(B)}$ however do not vary naturally in the variable B; the most we can say is that for each function $f \in \mathrm{Set}(B, B')$ the following square is commutative:

$$\begin{array}{ccc} X^{B'} \times B & \xrightarrow{X^f \times B} & X^B \times B \\ {\scriptstyle X^{B'} \times f}\downarrow & & \downarrow{\scriptstyle \epsilon} \\ X^{B'} \times B' & \xrightarrow[\epsilon]{} & X. \end{array} \tag{1.3}$$

This relation does not resemble naturality so much, but it can be easily deduced from the requirement that the adjunction isomorphisms (1.1) are natural in the variable B. In fact, such naturality imposes the

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commutation of the square

$$\begin{array}{ccc} \mathrm{Set}(A, X^{B'}) & \longrightarrow & \mathrm{Set}(A \times B', X) \\ {\scriptstyle \mathrm{Set}(A, X^f)}\downarrow & & \downarrow{\scriptstyle \mathrm{Set}(A \times f, X)} \\ \mathrm{Set}(A, X^B) & \longrightarrow & \mathrm{Set}(A \times B, X) \end{array} \tag{1.4}$$

for an arrow $f : B \to B'$ (the horizontal maps are the adjunction isomorphisms $\_ \mapsto \epsilon \circ (\_ \times B)$), and this in turn entails that we have equations

$$\epsilon_{X,(B')} \circ (u \times B') \circ (A \times f) = \epsilon_{X,(B)} \circ (X^f \circ u) \times B \tag{1.5}$$

$$\epsilon_{X,(B')} \circ (X^{B'} \times f) \circ (u \times B) = \epsilon_{X,(B)} \circ (X^f \times B) \circ (u \times B) \tag{1.6}$$

for every $u : A \to X^{B'}$. But since this is an equality for every such u, the functions $\epsilon_{X,(B')} \circ (X^{B'} \times f)$ and $\epsilon_{X,(B)} \circ (X^f \times B)$ must also be equal.

So it would seem that there is no way to frame the diagram above in the usual context of naturality for a transformation of functors. Fortunately, a suitable generalisation of naturality (a 'supernaturality' condition) encoding the above commutativity is available to describe this and other similar phenomena.

As already said, the correspondence $(B, B') \mapsto C^B \times B'$ is a functor with domain $\mathrm{Set}^{\mathrm{op}} \times \mathrm{Set}$; it turns out that these functors, where the domain is a product of a category with its opposite, support a notion of *di*naturality besides classical naturality. This notion is more suited to capturing the phenomenon we just described: in fact, most of the transformations that are canonical, depending on two variables $(C, C') \in \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$, but not natural, can be seen as dinatural.

Definition 1.1.1 (Dinatural transformation)**.** Let $\mathcal{C}, \mathcal{D}$ be two categories. Given two functors $P, Q : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$, a *dinatural transformation* $\alpha : P \Rightarrow Q$ consists of a family of arrows

$$\alpha_C : P(C, C) \to Q(C, C) \tag{1.7}$$

indexed by the objects of $\mathcal{C}$ and such that for any $f : C \to C'$ the following diagram commutes

$$\begin{array}{ccccc} P(C', C) & \xrightarrow{P(f,C)} & P(C, C) & \xrightarrow{\alpha_C} & Q(C, C) \\ {\scriptstyle P(C', f)}\downarrow & & & & \downarrow{\scriptstyle Q(C, f)} \\ P(C', C') & \xrightarrow[\alpha_{C'}]{} & Q(C', C') & \xrightarrow[Q(f, C')]{} & Q(C, C'). \end{array} \tag{1.8}$$

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Remark 1.1.2. The notion of dinaturality takes into account the fact that a functor $P : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ maps at the same time two 'terms' of the same 'type' $\mathcal{C}$, once covariantly in the second component and once contravariantly in the first. On arrows $f : C \to C'$ the functor P acts as follows:

$$\begin{array}{ccc} & P(C',C) & \\ {}^{P(C,f)}\swarrow & & \searrow^{P(C',f)} \\ P(C,C) & & P(C',C') \end{array} \tag{1.9}$$

Given two such functors, say $P, Q : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$, we can consider the two diagrams (1.9) and

$$\begin{array}{ccc} Q(C,C) & & Q(C',C') \\ {}_{Q(C,f)}\searrow & & \swarrow_{Q(f,c)} \\ & Q(C,C') & \end{array} \tag{1.10}$$

In the same way that a natural transformation $F \Rightarrow G$ can be seen as a family of maps that 'fill the gap' between $F(f)$ and $G(f)$ in a commutative square, a *dinatural* transformation between P and Q can be seen as a way to close the hexagonal diagram connecting the action on arrows of P to the action on arrows of Q:

$$\begin{array}{ccc} & P(C',C) & \\ {}^{P(C,f)}\swarrow & & \searrow^{P(C',f)} \\ P(C,C) & & P(C',C') \\ \vdots & & \vdots \\ Q(C,C) & & Q(C',C') \\ {}_{Q(C,f)}\searrow & & \swarrow_{Q(f,C')} \\ & Q(C,C') & \end{array} \tag{1.11}$$

This is precisely the diagram drawn in (1.8).

Remark 1.1.3. If we let P_C be the functor $(U,V) \mapsto C^U \times V$, the counit components $\epsilon_{C(B)} : P_C(B,B) \to C$ of the cartesian closed adjunction form a dinatural transformation $\epsilon : P_C \stackrel{..}{\Rightarrow} \Delta_C$, where Δ_C is the constant functor at C.

Such dinatural transformations, having constant codomain, deserve a special name.

Definition 1.1.4 ((Co)wedge)**.** Let $P : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{D}$ be a functor;

WC1. A *wedge* for P is a dinatural transformation $\Delta_D \Rightarrow P$ from the constant functor on the object $D \in \mathcal{D}$ (we often denote such a constant functor simply by the name of the constant, $D : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{D}$) defined by the rules $(C, C') \mapsto D$, $(f, f') \mapsto \text{id}_D$.

WC2. Dually, a *cowedge* for P as above is a dinatural transformation $P \Rightarrow \Delta_D$ having codomain the constant functor on the object $D \in \mathcal{D}$.

Remark 1.1.5. Wedges for a fixed functor P as above form the class of objects of a category $\text{Wd}(P)$, where a morphism of wedges is a morphism between their domains that makes an obvious triangle commute; given two wedges $\alpha : D \Rightarrow P$ and $\alpha' : D' \Rightarrow P$, a morphism of wedges consists of an arrow $u : D \to D'$ such that the triangle

$$\begin{array}{ccc} D & \xrightarrow{\quad u \quad} & D' \\ & {\scriptstyle \alpha_{CC}} \searrow \quad \swarrow {\scriptstyle \alpha'_{CC}} & \\ & P(C, C) & \end{array} \tag{1.12}$$

is commutative for every component α_{CC} and α'_{CC}. (Note the role of quantifiers: the same u makes (1.12) commute for *every* component of the wedges.)

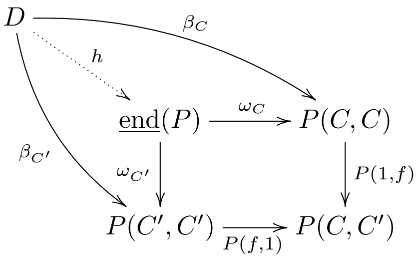
Dually, there is a category $\text{Cwd}(P)$ of cowedges for P, where morphisms of cowedges are morphisms between codomains (of course there is a relation between the two categories: cowedges for P coincide with the opposite category of wedges for the opposite functor).

We now define the end of P as a terminal object in $\text{Wd}(P)$, and the coend as an initial object in $\text{Cwd}(P)$.

Definition 1.1.6 ((Co)end)**.** Let $P : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{D}$ be a functor.

- The *end* of P consists of a terminal wedge $\omega : \underline{\text{end}}(P) \Rightarrow P$; the object $\underline{\text{end}}(P) \in \mathcal{D}$ itself is often called the *end* of the functor.
- Dually, the *coend* of P as above consists of an initial cowedge $\alpha : P \Rightarrow \underline{\text{coend}}(P)$; similarly, the object $\underline{\text{coend}}(P)$ itself is often called the coend of P.

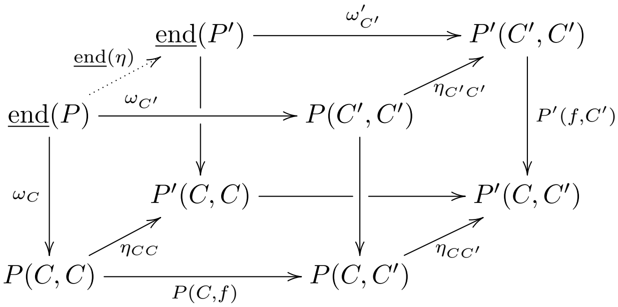
Spelled out explicitly, the universality requirement means that for any other wedge $\beta : D \Rightarrow P$ the diagram



(1.13)

commutes for a unique arrow $h : D \to \underline{\mathrm{end}}(P)$, for every arrow $f : C \to C'$. Note again the role of quantifiers: the arrow h is the same for every component of the wedge. A dual diagram can be depicted for the coend of P.

Remark 1.1.7 (Functoriality of ends)**.** Given a natural transformation $\eta : P \Rightarrow P'$ between functors $P, P' : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ there is an induced arrow $\underline{\mathrm{end}}(\eta) : \underline{\mathrm{end}}(P) \to \underline{\mathrm{end}}(P')$ between their ends, as depicted in the diagram



(1.14)

When all ends exist, sending a functor P into its end $\underline{\mathrm{end}}(P)$ is a (covariant) functor $\underline{\mathrm{end}} : \mathrm{Cat}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathcal{D}) \to \mathcal{D}$. $\underline{\mathrm{end}}$ preserves composition of morphisms, because the usual uniqueness argument using the universal property applies, and the arrow $\underline{\mathrm{end}}(\eta) \circ \underline{\mathrm{end}}(\eta')$ must coincide with $\underline{\mathrm{end}}(\eta \circ \eta')$ in a suitable pasting of cubes. Similarly, the unique arrow induced by $\mathrm{id}_P : P \Rightarrow P$ must be the identity of $\underline{\mathrm{end}}(P)$.

1.1.2 Extranaturality

A slightly less general, but better behaved[1] notion of supernaturality, which allows us again to define (co)wedges and thus (co)ends, is available: the notion is called *extra*naturality and it was introduced in [EK66b].

Definition 1.1.8 (Extranatural transformation)**.** Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be categories, and P, Q be functors

$$P : \mathcal{A} \times \mathcal{B}^{\mathrm{op}} \times \mathcal{B} \to \mathcal{D},$$
$$Q : \mathcal{A} \times \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}.$$

An *extranatural transformation* $\alpha : P \overset{..}{\Rightarrow} Q$ consists of a collection of arrows

$$\alpha_{ABC} : P(A, B, B) \longrightarrow Q(A, C, C) \tag{1.15}$$

indexed by triples of objects in $\mathcal{A} \times \mathcal{B} \times \mathcal{C}$ such that the following hexagonal diagram commutes for every triple of arrows $f : A \to A'$, $g : B \to B'$, $h : C \to C'$, all chosen from the appropriate domains:

$$\begin{array}{ccccc} P(A, B', B) & \xrightarrow{P(f,B',g)} & P(A', B', B') & \xrightarrow{\alpha_{A'B'C}} & Q(A', C, C) \\ {\scriptstyle P(A,g,B)}\downarrow & & & & \downarrow{\scriptstyle Q(A',C,h)} \\ P(A, B, B) & \xrightarrow[\alpha_{ABC'}]{} & Q(A, C', C') & \xrightarrow[Q(f,h,C')]{} & Q(A', C, C'). \end{array} \tag{1.16}$$

Notice how this commutative hexagon can be equivalently described as the juxtaposition of three distinguished commutative squares, depicted in [EK66b]: the three can be obtained by letting f and h, f and g, or g and h respectively be identities in the former diagram, which thus collapses to

$$\begin{array}{ccc} P(A, B, B) & \xrightarrow{P(f,B,B)} & P(A', B, B) \\ {\scriptstyle \alpha_{ABC}}\downarrow & & \downarrow{\scriptstyle \alpha_{A'BC}} \\ Q(A, C, C) & \xrightarrow[Q(f,C,C)]{} & Q(A', C, C) \end{array} \qquad \begin{array}{ccc} P(A, B', B) & \xrightarrow{P(A,B',g)} & P(A, B', B') \\ {\scriptstyle P(A,g,B)}\downarrow & & \downarrow{\scriptstyle \alpha_{AB'C}} \\ P(A, B, B) & \xrightarrow[\alpha_{ABC}]{} & Q(A, C, C) \end{array}$$

[1] We say *better behaved* since extranaturality admits a graphical calculus translating commutativity requirements into the requirement that certain string diagrams can be deformed one into the other.

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$$\begin{array}{ccc} P(A,B,B) & \xrightarrow{\alpha_{ABC}} & Q(A,C,C) \\ {\scriptstyle \alpha_{ABC'}}\downarrow & & \downarrow{\scriptstyle Q(A,C,h)} \\ Q(A,C',C') & \xrightarrow[Q(A,h,C')]{} & Q(A,C,C') \end{array} \tag{1.17}$$

Definition 1.1.9 (Mute functor)**.** Let $n \geq 1$ and $F : \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \to \mathcal{B}$ be a functor; we say that F is *mute in its* i*th component* if F factors as

$$\begin{array}{ccc} \prod_i \mathcal{A}_i & \longrightarrow & \mathcal{B} \\ & \searrow & \nearrow {\scriptstyle \bar{F}} \\ & \prod_{k \neq i} \mathcal{A}_k & \end{array} \tag{1.18}$$

where π_i is the projection canceling out the ith factor of the product.

Remark 1.1.10. We can again define (co)wedges in this setting: if $\mathcal{B} = \mathcal{C}$ and in $P(A,B,B) \to Q(A,C,C)$ the functor P is the constant functor on $D \in \mathcal{D}$, and $Q(A,C,C) = \bar{Q}(C,C)$ is mute in A, then we get a wedge condition for $D \Rightarrow Q$; dually we obtain a cowedge condition for $P(B,B) \to Q(A,B,B) \equiv D'$ for all A, B, C.

It is worth mentioning that an extranatural transformation contains more information than a dinatural one, since in 1.1.8 we are given arrows

$$F(B,B) \xrightarrow{\alpha_{BB'}} G(B',B') \tag{1.19}$$

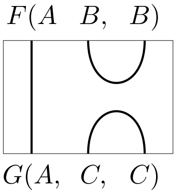
that are simultaneously a cowedge in B for each B', and a wedge in B' for all $B, B' \in \mathcal{B}$. We shall see in a while that extranaturality can be obtained as a special case of dinaturality.

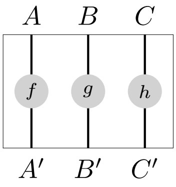
Both dinatural and extranatural transformations give rise to the same notion of (co)end, defined as a universal (co)wedge for a bifunctor $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$. (More formally, the notion of dinatural (co)wedge is indistinguishable from the notion of extranatural (co)wedge, and thus the two give rise to the same notion of (co)end.)

We prefer extranaturality for a variety of reasons:

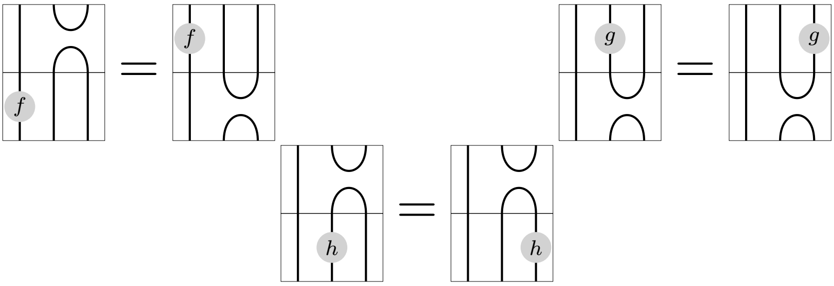
- it is less general (see Proposition 1.1.13), but it still makes (co)ends available;
- it gives rise to a fairly intuitive *graphical calculus* (see 1.1.11), and moreover it behaves better under composition (see Exercise 1.4);
- extranaturality is the correct notion in the enriched categorical setting (see 4.3.7 and the caveat right after).

Definition 1.1.11 (Graphical calculus for extranaturality)**.** ☉☉ The graphical calculus for extranatural transformations depicts the components α_{ABC}, and arrows $f : A \to A'$, $g : B \to B'$, $h : C \to C'$, respectively as planar diagrams such as

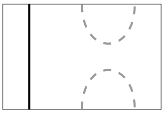




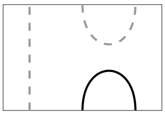
where wires are labelled by objects and must be thought of as oriented from top to bottom. The commutative squares of (1.17) become, in this representation, the following three string diagrams, whose equivalence is graphically obvious (the labels f, g, h are allowed to 'slide' along the wire they live on):

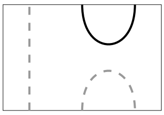


Remark 1.1.12. The notion of extranatural transformation can be specialised to encompass various other constructions. Simple naturality arises when F, G are both constant in their (co)wedge components, so the cap and cup in α_{ABC} vanish:



The wedge and cowedge conditions arise when either of F, G is constant, so that the straight line and one of the cup or the cap in α_{ABC} vanish:





All the other mixed situations (a wedge-cowedge condition, naturality and a wedge, and others that do not have a specified name) admit a graphical representation of the same sort, and follow similar graphical rules of juxtaposition, when the boundaries of their associated cells agree in shape in the obvious sense.

All extranatural transformations can be obtained as particular cases of dinatural transformations; on the contrary, there are dinatural transformations that are not extranatural. An example is given in Exercise 1.5.

Proposition 1.1.13. Extranatural transformations are particular kinds of dinatural transformations.

Proof (due to T. Trimble) Given functors $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \times \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \times \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$, set $\mathcal{A} = \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}$, and form two new functors $F', G' : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{D}$ by taking the composites

$$F' = (\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \times \mathcal{C}) \times (\mathcal{C} \times \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}) \xrightarrow{\mathrm{proj}} \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \times \mathcal{C} \xrightarrow{F} \mathcal{D}$$
$$(X', Y', Z'; X, Y, Z) \longmapsto (X', X, Y') \overset{F}{\mapsto} F(X', X, Y')$$
$$G' = (\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \times \mathcal{C}) \times (\mathcal{C} \times \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}) \xrightarrow{\mathrm{proj}'} \mathcal{C} \times \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \xrightarrow{G} \mathcal{D}$$
$$(X', Y', Z'; X, Y, Z) \mapsto (Y', Z', Z) \overset{G}{\mapsto} G(Y', Z', Z).$$

Now let us put $A' = (X', Y', Z')$ and $A = (X, Y, Z)$, considered as objects in $\mathcal{A}$. An arrow $\varphi : A' \to A$ in $\mathcal{A}$ thus amounts to a triple of arrows $f : X' \to X$, $g : Y \to Y'$, $h : Z \to Z'$, all in $\mathcal{C}$. Following the instructions above, we have $F'(A', A) = F(X', X, Y')$ and $G(A', A) = G(Y', Z', Z)$. Now if we write down a dinaturality hexagon for $\alpha : F' \Rightarrow G'$, we get a diagram of shape

$$\begin{array}{ccccc} F'(A, A') & \xrightarrow{F'(1,\varphi)} & F'(A, A) & \xrightarrow{\alpha_A} & G'(A, A) \\ {\scriptstyle F(\varphi,1)}\downarrow & & & & \downarrow{\scriptstyle G'(\varphi,1)} \\ F'(A', A') & \xrightarrow[\alpha_{A'}]{} & G'(A', A') & \xrightarrow[G'(1,\varphi)]{} & G(A', A) \end{array} \tag{1.20}$$