Preface

This tract is perhaps written in a more didactic manner than others in the series, but its motivating purpose is largely the same: to acquaint its readers with important aspects of thought within the philosophy of mathematics. I would like to thank Jeremy Avigad, Kathleen Cook, Rachael Driver, Anil Gupta, Julliette Kennedy, Michael Liston, Penelope Maddy, and Stewart Shapiro for helpful comments on these topics.

1 Suggestions from the Symbols

> But now the way seems open to us, still further to generalize the Abstract Geometry, with the help of suggestions arising from the symbols themselves, using the words point, line, etc., in a proper sense consistent therewith. . . . Two questions naturally arise: (1), Is there any geometrical utility in this extension? (2), Is it legitimate to use the postulated properties of the abstract points, lines, etc., in order to prove relations existing among the real points, lines., etc., that is, relations which can be stated without any reference to the abstract elements? . . . [And] it may be said, briefly, that experience has amply shewn that the gain in the generality of the statements of geometrical fact, and the increased power of recognizing the properties of a geometrical figure, enormously outweigh the initial feeling of artificiality and abstractness. . . . [T]he introduction of [extra] elements may well have assisted the constructive faculty [of ingenuity]; that this may happen is, indeed, one of the discoveries of the history of reasoning.
>
> H. F. Baker (1923, pp. 143–44)

Mathematicians and philosophers hope that their proposals will remain unblemished forever: permanent monuments of truth chiseled from the impermeable rock of a priori necessity. They can cheerfully concede that future generations may place their results within wider frameworks that seem more accommodating or practical, for any Gauss or Kant should acknowledge that their specific findings can be fit within conceptual contexts that they had not anticipated. Nevertheless, such authors can remain confident that their original discoveries will remain as unperishable truths even within these enlarged surroundings. “A diamond is forever,” runs the old jewelry advertisement, and the same assurance applies to the accumulated gems of mathematical and philosophical discovery. Physicists, biologists, economists, and other human creatures must recognize that everything they propose will be eventually overturned, but mathematicians and philosophers need not harbor comparable fears, as long as they have remained properly diligent in their methodological rigors.

But does this vein of thinking constitute a self-flattering illusion? Jurist Oliver Wendell Holmes Jr. thought so: “[L]ogical method and form flatter that longing for certainty and for repose which is in every human mind. But certainty generally is illusion, and repose is not the destiny of man" (1897, p. 3).

The purpose of this Element is to survey some of the challenges that the natural *enlargements of domain* to which mathematics has been continually subjected pose with respect to this conception of apriorist necessity. (These foundational adjustments are often labeled as "changes in setting.") Operationally, the developmental pressures that prompt these shifts come into play when mathematicians attempt to establish a deductive pathway linking locations A and B and discover that their journey will be greatly facilitated if they are allowed to travel through intermediate locations C, lying outside of the boundaries in which the task was originally posed.

An early example arose as sixteenth-century mathematicians attempted to find real number solutions to cubic equations such as $x^3 + px + q = 0$. (Prizes were awarded to contestants who could produce such answers in the quickest time.) In 1545, through brute symbolic manipulations, G. Cardano arrived at the formula we would write today as:

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

In suitable circumstances, this supplies the desired roots. However, the two cubic roots in Cardano's formula seemingly designate "impossible" (= complex) values in many cases, even if these "impossibilities" eventually cancel out when added together. These "impossible values" comprise examples of the useful, out-of-country locations featured in our geographical analogy. We will survey a range of cases of this sort (a good history is provided by Katz and Parshall 2014.)

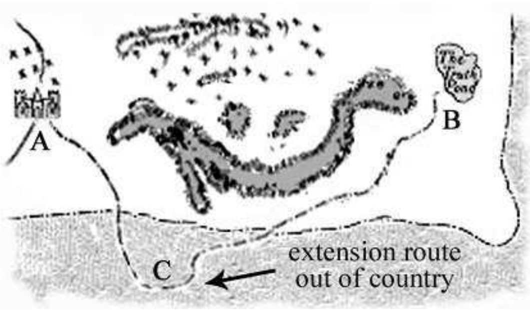

Figure 1. Out-of-country elements

As it happens, most of us have become inured to negative and imaginary numbers from high school algebra, viewing them as unproblematic extracts from "the world of mathematics," without realizing that long into the nineteenth century both items were warily regarded as computational instrumentalities lacking coherent significance. Up to the time of Kant (more or less), the traditional realms of "geometry" and "number" were regarded as the only domains in which the synthetic a priori reasonings of the mathematical sciences can be reliably set. To disarm the present-day complacencies bred by subsequent familiarity, we must first recover the intellectual shock that nineteenth-century observers often expressed when they were first presented with some of these "innovative" changes. For many of us, the strange "extension elements" that mathematicians added to regular Euclidean geometry in the first part of the nineteenth century[1] can still serve this purpose if we have not studied the modern subject of "algebraic geometry," in which these novel ingredients are now central ingredients. Absent such a background, the resulting claims will likely strike many of us as bizarre, viz. the proposition that two apparently non-overlapping circles secretly intersect in four points: two of them regular imaginary and two additional "circular points" on the line at infinity. (We will review the motivating rationale for these weird claims in a moment.) In 1883, one of the prominent developers of these dark arts, the mathematician Arthur Cayley, called for a philosophical examination of their rationale:

> [T]he notion which is really the fundamental one (and I cannot too strongly emphasize the assertion) underlying and pervading the whole of modern analysis and geometry [is] that of imaginary magnitude in analysis and of imaginary . . . points and figures in geometry. This [topic] has not been, so far as I am aware, a subject of philosophical discussion or inquiry . . . [E]ven [if our final] conclusion were that the notion belongs to mere technical mathematics, or has reference to nonentities in regard to which no science is possible, still it seems to me that as a subject of philosophical discussion the notion ought not to be this ignored; it should at least be shown that there is a right to ignore it. (1889, p. 433)

And the answer Cayley himself suggests sounds disconcertedly mystical in its invocation of Plato's cave:

> That we cannot "conceive" [of "purely imaginary objects"] depends on the meaning which we attach to the word conceive. I would myself say that the purely imaginary objects are the only realities, the *ὄντως ὄντα* ("the realities that really exist"), in regard to which the corresponding physical objects are

[1] This period is frequently labeled as the "projective geometry revolution." It should not be confused with the non-Euclidean geometry that became popular later, which actually raises fewer methodological puzzles in its wake.

> as the shadows in the cave; and it is only by means of them that we are able to deny the existence of a corresponding physical object; if there is no conception of straightness, then it is meaningless to deny the existence of a perfectly straight line. (1899, p. 433)

Soon thereafter, a range of contemporaneous philosophers (e.g., Ernst Cassirer and Bertrand Russell) actively engaged with Cayley's concerns, often in relatively unsatisfactory ways. But independently of their academic proposals, virtually every working mathematician of the late nineteenth century needed to ponder these methodological issues in some manner or other, if only to realign their own investigative compasses along the axes of fruitful inquiry that were dramatically restructuring the subjects in which they worked. In this Element, we will particularly focus on the intriguing methodological suggestions found in the pithy remarks offered on these topics by the great nineteenth-century algebraist Richard Dedekind, whose methodological shadow has loomed over mathematical practice ever since.[2] Some of his central themes have been largely overlooked by his modern admirers, despite the fact that they paint a portrait of the mathematical enterprise that remains entirely pertinent to our own era – or so this brief tract will argue.

C. F. Gauss designated mathematics as "the queen of the sciences" (displacing theology from its former pride of place), and by equal rights, the *philosophy* of mathematics ought to perch upon a comparable throne within philosophy as well. And that was the prestige with which philosophers of earlier times accorded its methodological concerns. Today, however, the subject has lost much of its former allure, and academic consideration has largely thinned into wan disquisitions on "naturalism" and "ontological commitment." As a result, the puzzles of innovative practices have become relegated to the sidelines of specialized concern, bearing little anticipated relevance to the central concerns of language, metaphysics, or the wider stretches of science. This Element will argue that this demotion is a mistake; an adequate appreciation of the motivational factors that drive mathematics to continually reshape old domains into considerably altered configurations ought to remain a central ingredient within our attempts to gauge the conceptual capabilities of human thought more generally.

Working mathematicians, of course, cannot afford to ignore the reconstitutive adjustments that continually redirect their disciplines in unexpected directions, for their academic standing may depend upon their ability to convince their colleagues that their innovations represent "the right way to proceed." However, a distaste for the disputes about "abstract objects" and so forth that dominate

[2] Emmy Noether: "Es steht alles schon bei Dedekind."

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current philosophical discussion has induced a profound *horror philosophiae* within mathematical circles, which frequently invoke simple formalist excuses (the "if-thenism" of Section 5) that allow them to beg off "waxing philosophical" in a manner they distrust.

Unfortunately, the considerations that guide research within modern mathematics have become forbiddingly technical, and an adequate mastery of their motivating threads is hard to obtain. To evade these pedagogical obstacles, this Element will largely concentrate on an assortment of easier-to-explain nineteenth-century adjustments in which the winds of innovation altered traditional mathematical landscapes substantially. An excellent starting point lies with those funny points of non-intersecting "intersection" mentioned earlier (which also represents one of the central cases that Cayley worried about).

The main impulse came from algebra. Descartes's innovations within what we now call "Cartesian geometry" forged unexpected pathways between theorems that could not be obtained through traditional Euclidean proof techniques. For example, ellipses, parabolas, and hyperbolas strike us as rather similar in their animating behaviors, but the Euclidean proofs required to establish that these facts differ significantly. In contrast, the same relationships can be established within Cartesian geometry by calculations directed to their common equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, if we are allowed to factor this expression into expressions that lack any obvious significance. But how can we trust a proof if we do not understand what its intermediate steps mean? These considerations prompted synthetic geometers such as J.-V. Poncelet to wonder if similar (yet intelligible) pathways of easy reasoning could not be established as proper to geometry if its internal dominions were extended through defensible policies for extending a preexistent domain. Indeed, supplementary "points at infinity" had already become familiar as the "horizons" and "vanishing points" within a perspective drawing.

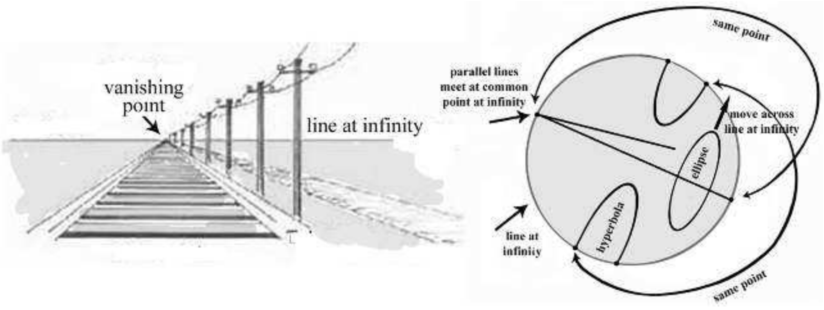

Figure 2. Points at infinity

Poncelet realized that these supplementary objects could be harnessed to significant inferential advantage if we allow ourselves to reason about these "lines and points at infinity" as if they represented regular Euclidean ingredients. From this extended point of view, the mysteriously similar behaviors of ellipses and hyperbolas can be explicated by simply dragging an ellipse across the line at infinity until it appears as if it has become split into two pieces.[3] By allowing parallel lines to intersect at such "points of infinity," we can likewise avoid annoying distinctions in our proofs between lines that cross somewhere and those that do not.

Allied pathways of improved reasoning similarly rationalize the strange "imaginary points" mentioned previously, for reasons that I will sketch shortly. But do we not risk spoiling the a priori certainties of the traditional Euclidean realm by rashly introducing these "extension element" supplements? It would likewise make algebraic calculations simpler if we could assign a factor such as "6/0" a convenient numerical value, but it proves impossible to do so without opening a door to harmful contradictions (i.e., the mere acceptance that "6/0" possesses a value immediately allows one to prove that "0 = 1"). Poncelet plainly requires some more sophisticated form of methodological justification for his innovations than the crudely pragmatic "They allow me to reach nice results quickly." Over the course of this Element, we will examine a succession of proposals to this purpose. We can begin with Poncelet's own justificatory offering, based upon a principle that he dubbed "the permanence of mathematical relations" (other authors call it "persistence of form"):

> Is it not evident that if, keeping the same given things, one can vary the primitive figure by insensible degrees by imposing on certain parts of the figure a continuous but otherwise arbitrary movement, is it not evident that the properties and relations found for the first system, remain applicable to successive states of the system, provided always that one has regard for certain particular modifications that may intervene, as when certain quantities vanish or change their sense or sign, etc., modifications which it will always be easy to recognize a priori and by infallible rules? . . . Now this principle, regarded as an axiom by the wisest mathematicians, one can call the principle or law of continuity for mathematical relationships involving abstract and depicted magnitudes. (1822, p. 19)

[3] Florian Cajori (1919, p. 62) characterizes the disadvantages of traditional Euclidean geometry as follows:

> The principal characteristics of the ancient geometry are:
>
> (1) A wonderful clearness and definiteness of its concepts and an almost perfect logical rigor of its conclusions.
> (2) A complete want of general principles and methods.

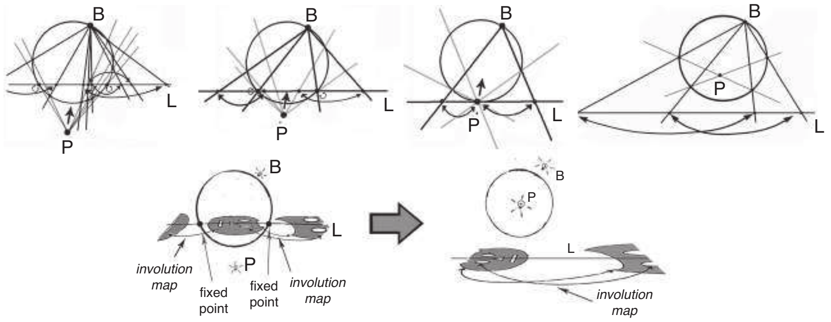

Figure 3. Pole and polar movement

Here is how these notions operate in the context of our "imaginary" geometrical points. Projective geometry asks how images adjust when an originating slide (say, a picture of a cat's head inscribed on the sides of a sphere C) is projected onto varying screens. As this occurs. the ears, mouth, and so forth will distort considerably as the image plane L is moved to different positions relative to a lamp B. However, certain abstract characteristics of the image must remain preserved within all of these placements. (Otherwise, we would not be able to recognize that "it's the same cat" throughout.) The projective geometers discovered that this "invariant" could be explained in terms of a geometrical relationship called a "cross-ratio." In Figure 3, I have tried to illustrate this construct in the upper sequence of diagrams, although the exact details are not important for our purposes.

We carve out a two-dimensional slice of our arrangements and consider how the projected image appears on our "screen" (the line L). By allowing light to travel backward across the line at infinity (!), two projected cat images will always appear on L. When a complete cat head fits inside the circle C connecting to our lamp B, we can locate an important exterior point P called the *pole* of the construction by drawing tangents from the two places where C intersects L (which is then called the *polar* of P). We can then correlate the respective parts of our two cat heads by a suitable mapping **m** called an *involution*. The two positions where L intersects C constitute *fixed points* of **m** in the sense that **m** maps these positions to themselves as self-corresponding. The cross-ratio invariant we seek can then be explicated in terms of the invariant manner in which the correlated cat features cluster together around these two fixed points. (This pattern represents a generalization of a "harmonic division" within traditional geometry.) As a result, the two fixed points represent the central "controlling points" around which the rest of the construction arranges itself.

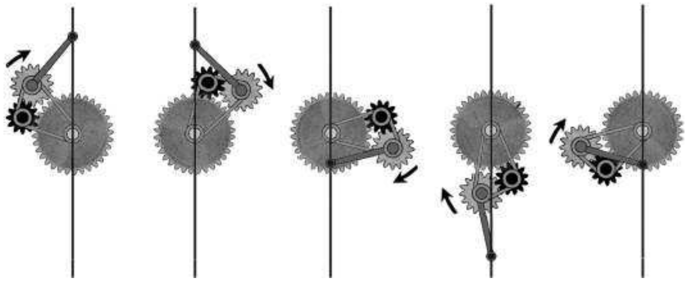

Figure 4. Machine movement

What happens if we now push the pole point P toward the interior of C? In particular, what happens to the polar line L? It will gradually move outward until it passes through a transition stage where L is tangent to C and its two fixed points coincide. Pushing P fully inside C, we reach an altered scenario in which the polar line L now lies outside of C, yet a similar involution pairing **m** between the cat parts remains well-defined (i.e., their cross-ratio remains the same). But **m**'s controlling fixed points have apparently vanished from the scene.

Or have they merely become invisible? Here's where Poncelet's "permanence of relationships" principle enters the story. Examine our successive P/L drawings as if they comprise successive frames within a motion picture film. The resulting montage supplies an evolving picture of integrated movement, much like we would witness in a film of an actual cycling mechanism, such as the Cardan gearing illustrated in Figure 4.

We previously noted that the original fixed points of the mapping **m** gradually move closer together until they fuse and seemingly disappear. Poncelet's principle contends that this "disappearance" is only apparent, because all of the other pole/polar/involution relationships remain intact, albeit altered in appearance. Our two fixed points have merely become "imaginary" by moving out of the plane of our paper, explaining why they can no longer be readily pictured within a conventional visual representation. Nonetheless, these imaginary fixed points continue to "control" the rest of the associated pattern (operating "from an astral plane" as it were). These same supplementary points also supply the mysterious "intersections" between non-overlapping circles to which I earlier alluded. However, a proper justification of their utility in Poncelet's manner requires consideration of the "persisting relationships" evident within a sequence of machine-like movements of the type illustrated.

By successively enlarging the dominions of traditional geometry in this manner, nineteenth-century geometers felt that they had quasi-inductively stumbled onto some hidden Platonic reality that renders the surface relationships of traditional geometry coherent. In 1832, the German geometer Jakob Steiner declared:

> The present work contains the final results of a prolonged search for fundamental spatial properties which contain the germ of all theorems, porisms and problems of geometry so freely bequeathed to us by the past and the present... It must be possible to find for this host of disconnected properties a leading thread and common root that would give us a comprehensive and clear overview of the theorems and better insight into their distinguishing features and mutual relationships. . . . By a proper appropriation of a few fundamental relations one becomes master of the whole subject; order takes the place of chaos, one beholds how all parts fit naturally into each other, and arrange themselves serially in the most beautiful order, and how related parts combine into well-defined groups. In this manner one arrives, as it were, at the elements, which nature herself employs in order to endow figures with numberless properties with the utmost economy and simplicity. (1832, p. 315)

In 1857 the president of Harvard, Thomas Hill, rhapsodized similarly:

> The conception of the inconceivable [imaginary], this measurement of what not only does not, but cannot exist, is one of the finest achievements of the human intellect. No one can deny that such imaginings are indeed imaginary. But they lead to results grander than any which flow from the imagination of the poet. The imaginary calculus is one of the master keys to physical science. These realms of the inconceivable afford in many places our only mode of passage to the domains of positive knowledge. Light itself lay in darkness until this imaginary calculus threw light upon light. And in all modern researches into electricity, magnetism, and heat, and other subtle physical inquiries, these are the most powerful instruments. (1857, p. 265)

And these same advantages accrue to the even stranger schemes and divisors characteristic of modern algebraic geometry:

> Our examples show the surprisingly wide range of possible behavior . . ., and the apparent jungle of possibilities leads to a basic question: Where are the nice theorems?
>
> A fundamental truth [then] emerges: to get nice theorems, algebraic curves must be given enough living space. For example, important things can happen at infinity, and points at infinity are beyond the reach of the real plane. We use a squeezing formula to shrink the entire plane down to a disk, allowing us to view everything in it. This picture leads to adjoining points at infinity, and in one stroke all sorts of exceptions then melt away. We [will] enhance the reader's intuition through pictures showing what some everyday curves look like after squeezing them into a disk.

> Continuing [our] quest for nice theorems, [we] once again [find that] the answer lies in giving algebraic curves additional living space – in this case we expand from the real numbers to the complex. Working over them, together with points added at infinity, we arrive at one of the major highlights of the book, Bézout's theorem. This is one of the most underappreciated theorems in mathematics, and it represents an outstandingly beautiful generalization of the Fundamental Theorem of Algebra. (Kendig 2010, p. ix)

On the other hand, a large number of later mathematicians were troubled by the mystical forms of Platonic appeal that such assertions invoke:

> [Otto Hesse regarded Steiner's later period] as marked by his struggle with the imaginary, or as Steiner liked to say, his quest to seek out those "ghosts" that hide their truths in a strange geometrical netherworld.[4] (Rowe 2018, p. 64)

In later sections, we will review how subsequent methodologists attempted to convert Poncelet's worthy (yet dodgy) appeals to the "persistence of relationships" into more rigorous forms of methodological justification.

But we should also observe that such "changes in setting" are not completely irrevocable. Felix Klein comments upon the worthy topics left behind:

> In one respect, of course, the Plücker formulas, in spite of their great generality, do leave some problems open: they yield nothing about the separation of the real from the imaginary. Even though abstract thought was indifferent to these questions for decades, they are still of the greatest interest to those who seek the true geometric shape of the varieties. It must be regarded as an aberration of modern geometry that the importance of this question is everywhere denied. . . . Plücker was no "projectivist" in the true sense. In the style of the old geometers of the 18th century he clung to the concrete, investigating [matters] . . . all of whose significance vanishes from the purely projective viewpoint.[5] (1926, p. 114)

Indeed, those old problems have reemerged within the modern context of robotics, in which we need to compute the locations where an automaton is likely to bump into the tables and chairs around it. Learning about their imaginary intersection points in the projective manner is not very helpful in this context. (More correctly, the projective extension elements may still prove useful in these pursuits, but only as halfway houses to the results we really need.)

As remarked earlier, most of us now regard the employment of complex and negative numbers within algebra as "old hat" and not particularly demanding of

[4] Compare (Klein 1908, p. 187): "To Steiner, imaginary quantities were ghosts, which made their effect felt in some way from a higher world without our being able to gain a clear notion of their existence."

[5] The formulas cited do not distinguish the real points of intersection between two figures from their "imaginary" crossings.