# FINITE SIMPLE GROUPS AND FUSION SYSTEMS

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This expository paper is taken from a series of four talks given at the conference Groups St Andrews in Birmingham 2017, held in August of 2017. The goal of those talks was to give the audience some insight into an ongoing program to, first, classify a certain class of simple 2-fusion systems, and then, second, to use the result on fusion systems to simplify the proof of the theorem classifying the finite simple groups (CFSG). But since the talks were delivered to a general audience of group theorists, most of the presentation was devoted to supplying background. The same is true of this article, where the program does not formally make an appearance until fairly late in the game.

Thus we'll begin with an introduction to the basic theory of fusion systems. Then we give an overview of the proof of that part of the CFSG devoted to the groups of component type, after which we discuss how to translate that proof into the category of 2-fusion systems, and indicate some advantages that accrue from that translation. We also describe some other changes to the original proof of the CFSG that are part of the program.

Our basic reference on fusion systems is [2], although [7] also supplies a good introduction to the subject. Our basic reference on finite groups is [1]. For a more detailed discussion of the proof of the CFSG see [3].

## Fusion systems

Let $p$ be a prime and $S$ a finite $p$-group. A *fusion system* on $S$ is a category $\mathcal{F}$ whose objects are the subgroups of $S$ and, for subgroups $P, Q$ of $S$, the set $\hom_{\mathcal{F}}(P, Q)$ of morphisms from $P$ to $Q$ is a set of injective group homomorphisms of $P$ into $Q$, and that set satisfies two weak axioms:

(1) If $s \in S$ with $P^s \leq Q$ then the conjugation map $c_s : P \to Q$ is a morphism.

(2) If $\phi : P \to Q$ is a morphism, then so is $\phi : P \to P\phi$ and $\phi^{-1} : P\phi \to P$.

Call $S$ the *Sylow group* of $\mathcal{F}$.

**Example 1.1** Let $G$ be a finite group, $S \in Syl_p(G)$, and $\mathcal{F}_S(G)$ the fusion system on $S$ whose morphisms are induced via conjugation in $G$. Call $\mathcal{F}_S(G)$ the *p-fusion system* of $G$.

We are primarily interested in saturated fusion systems. A fusion system $\mathcal{F}$ is *saturated* if it satisfies two more axioms, that can be easily seen to hold in Example 1.1 using Sylow's Theorem. See [2] for the axioms.

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A saturated system $\mathcal{F}$ is *exotic* if it is the fusion system of no finite group; there exist exotic systems, and indeed for $p$ odd they seem to proliferate. However 2-fusion systems seem to be more well behaved.

## Luis Puig, modular representation theory, and algebraic topology

The notion of a fusion system and much of the basic theory of fusion systems is due to Luis Puig, except that Puig uses different terminology and notation; see for example [13]. Puig's primary interest is modular representation theory. I'm using notation and terminology due to some algebraic topologists, particular Broto, Levi, and Oliver in [6], since I learned the subject from their papers, and their terminology has by now become standard.

In short, fusion systems originally arose in the context of modular representation theory, and remain of significant interest in that area. And of course algebraic topologists contribute to, and make use of, the theory of fusion systems; for example the Martino-Priddy Conjecture (now a theorem [11], [12]) says the $p$-completed classifying spaces of a pair of finite groups are of the same homotopy type precisely when their $p$-fusion systems are isomorphic.

But I'm going to say no more about the role of fusion systems in representation theory and topology, and instead focus on the relationship between fusion systems and local finite group theory. However it should be noted that one of the advantages of the fusion system approach is that it draws on both topology and algebra.

## A functor

Let $\mathcal{F}$ be a fusion system on $S$. If $\tilde{\mathcal{F}}$ is a fusion system on $\tilde{S}$ then a *morphism* from $\mathcal{F}$ to $\tilde{\mathcal{F}}$ is a group homomorphism $\alpha : S \to \tilde{S}$ such that $\alpha$ induces a map from morphisms of $\mathcal{F}$ to morphisms of $\tilde{\mathcal{F}}$. For example if $\alpha = \beta_{|S}$ for some homomorphism $\beta : G \to \tilde{G}$ then $\alpha : \mathcal{F}_S(G) \to \mathcal{F}_{\tilde{S}}(\tilde{G})$ is a morphism of fusion systems.

Indeed let $\mathfrak{G}$ be the category whose objects are pairs $(G, S)$ with $G$ a finite group and $S \in Syl_p(G)$, and with a *morphism* from $(G, S)$ to $(\tilde{G}, \tilde{S})$ a group homomorphism $\beta : G \to \tilde{G}$ with $S\beta \leq \tilde{S}$. Then we have a functor $(G, S) \mapsto \mathcal{F}_S(G)$ and $\beta \mapsto \beta_{|S}$ from $\mathfrak{G}$ to the category of saturated fusion systems. The game is to use this functor to translate notions involving finite groups to analogous notions concerning fusion systems, and to prove theorems in one of the two categories using an analogous theorem in the other category.

## A local theory of fusion systems

For $P \leq S$ the set $P^{\mathcal{F}}$ of *conjugates* of $P$ consists of the images $P\phi$, $\phi \in \hom_{\mathcal{F}}(P, S)$. In Example 1.1, $P^{\mathcal{F}}$ is the set of $G$-conjugates of $P$ contained in $S$.

Define $P$ to be *fully normalized*, *fully centralized* if for each $Q \in P^{\mathcal{F}}$, we have $|N_S(P)| \geq |N_S(Q)|$, $|C_S(P)| \geq |C_S(Q)|$, respectively. In Example 1.1, $P$ is fully normalized precisely when $N_S(P) \in Syl_p(N_G(P))$. Write $\mathcal{F}^f$ for the set of fully normalized subgroups of $S$.

The *local theory of finite groups* studies finite groups $G$ from the point of view of the local subgroups of $G$. Here a *p-local subgroup* of $G$ is the normalizer $N_G(P)$ of some nontrivial $p$-subgroup $P$ of $G$. What is the right notion of a local subsystem of $\mathcal{F}$?

Let $P \leq S$ and define $N_{\mathcal{F}}(P)$, $C_{\mathcal{F}}(P)$ to be the subfusion system $\mathcal{E}$ of $\mathcal{F}$ with Sylow group $T = N_S(P)$, $C_S(P)$, such that for $Q \leq T$, $\phi \in \hom_{\mathcal{F}}(Q,T)$ is an $\mathcal{E}$-morphism if and only if $\phi$ extends to $\varphi \in \hom_{\mathcal{F}}(PQ, PT)$ with $P\varphi = P$, $\varphi$ centralizing $P$, respectively.

In Example 1.1, if $P$ is fully normalized then $N_{\mathcal{F}}(P) = \mathcal{F}_{N_S(P)}(N_G(P))$.

The systems $N_{\mathcal{F}}(P)$ for $1 \neq P \in \mathcal{F}^f$, play the role of the local subsystems of $\mathcal{F}$. We want our local subsystems to be saturated; this follows from the following fundamental lemma of Puig:

**Lemma 1.2 (Puig)** *Let $\mathcal{F}$ be saturated and $P \in \mathcal{F}^f$. Then $N_{\mathcal{F}}(P)$ and $C_{\mathcal{F}}(P)$ are saturated.*

Define a subgroup $P$ of $S$ to be *normal* in $\mathcal{F}$, and write $P \trianglelefteq \mathcal{F}$, if $\mathcal{F} = N_{\mathcal{F}}(P)$. In Example 1.1 if $P \trianglelefteq G$ then $P \trianglelefteq \mathcal{F} = \mathcal{F}_S(G)$, but the converse is not in general true. For example let $G$ be a nonabelian finite simple group with abelian Sylow $p$-subgroup $S$ and $\mathcal{F} = \mathcal{F}_S(G)$. Then by Burnside's Fusion Theorem we have $\mathcal{F} = N_{\mathcal{F}}(S)$, so $S \trianglelefteq \mathcal{F}$ but of course $S$ is not normal in $G$.

The product of normal subgroups of $\mathcal{F}$ is normal in $\mathcal{F}$, so $\mathcal{F}$ has a largest normal subgroup, which we denote by $O_p(\mathcal{F})$.

A subgroup $P$ of $S$ is *centric* if for each $Q \in P^{\mathcal{F}}$ we have $C_S(Q) \leq Q$. In Example 1.1, $P$ is centric if and only if $P$ contains each $p$-element in $C_G(P)$. Write $\mathcal{F}^c$ for the set of centric subgroups of $S$.

Next $P$ is *radical* if $Inn(P) = O_p(Aut_{\mathcal{F}}(P))$. Write $\mathcal{F}^r$ for the set of radical subgroups and $\mathcal{F}^{frc}$ for the set of fully normalized, radical, centric subgroups.

**Remark 1.3** Let $\mathcal{F}$ be saturated.

(1) One of the axioms of saturation says that if $P \in \mathcal{F}^f$ then $Aut_S(P) \in Syl_p(Aut_{\mathcal{F}}(P))$.

(2) $S \in \mathcal{F}^{frc}$.

(3) If $P \in \mathcal{F}^{frc} - \{S\}$ then $Aut_{\mathcal{F}}(P)$ is not a $p$-group.

It is easy to see that $S \in \mathcal{F}^{fc}$, while $S$ is radical by (1). If $P \in \mathcal{F}^{frc}$ and $Aut_{\mathcal{F}}(P)$ is a $p$-group, then as $P$ is radical we have $Inn(P) = Aut_{\mathcal{F}}(P)$. But by (1), $Aut_S(P)$ is Sylow in $Aut_{\mathcal{F}}(P)$, so $Aut_S(P) = Inn(P)$. Therefore $N_S(P) = PC_S(P)$, and as $P$ is centric we have $C_S(P) \leq P$, so that $N_S(P) = P$, and hence $S = P$. That is (3) holds.

## Generation

Let $T \le S$ and $\Delta$ a set of morphisms between subgroups of $T$. The intersection of any collection of fusion systems on $T$ is again a fusion system on $T$. Thus there is a smallest fusion system on $T$ containing $\Delta$, which we denote by $\langle \Delta \rangle_T$. Call this system the *subsystem on $T$ generated by* $\Delta$.

**Theorem 1.4 (Alperin's Fusion Theorem)** *Assume $\mathcal{F}$ is saturated. Then $\mathcal{F} = \langle Aut_{\mathcal{F}}(R) : R \in \mathcal{F}^{frc} \rangle_S$.*

**Remark 1.5** Observe that if $\Xi$ is a set of representatives for the orbits of $S$ on $\mathcal{F}^{frc}$ then $\mathcal{F} = \langle Aut_{\mathcal{F}}(R) : R \in \Xi \rangle_S$.

**Example 1.6** Let $p = 2$ and $S = D_m$ be dihedral of order $m > 4$. Let's determine, up to isomorphism, the saturated fusion systems on $S$.

First $S$ has two conjugacy classes $E_i^S$, $i = 1, 2$ of 4-subgroups, with $E_1$ and $E_2$ fused in $Aut(S)$. Moreover for $P \le S$, $Aut(P)$ is not a 2-group if and only if $P \cong E_4$, in which case $Aut(P) = GL(P) \cong S_3$. It follows from Remark 1.3 that for $R \le S$ we have $R \in \mathcal{F}^{frc}$ if and only if $R = S$ or $R$ is a 4-group with $Aut_{\mathcal{F}}(R) = Aut(R)$, and with $R \in E_i^S$ for some $i$ in the last case. Hence by Remark 1.5, up to isomorphism there are three potential saturated fusion systems on $S$:

(1) $\mathcal{F}_0$ where $Aut_{\mathcal{F}}(E_i) = Aut_S(E_i) \cong \mathbf{Z}_2$ for $i = 1, 2$.

(2) $\mathcal{F}_1$ where $Aut_{\mathcal{F}}(E_1) = Aut(E_1) \cong S_3$ and $Aut_{\mathcal{F}}(E_2) = Aut_S(E_2) \cong \mathbf{Z}_2$.

(3) $\mathcal{F}_2$ where $Aut_{\mathcal{F}}(E_i) = Aut(E_i) \cong S_3$ for $i = 1, 2$.

For $0 \le j \le 2$ let $G_j$ be a finite group with $S \in Syl_2(G_j)$ such that $G_0 = S$, $G_1 \cong PGL_2(q_1)$, and $G_2 \cong L_2(q_2)$ for suitable odd $q_j$. Then $\mathcal{F}_j = \mathcal{F}_S(G_j)$, so $\mathcal{F}_j$ is saturated.

Notice this proof also shows that there are exactly four saturated fusion systems on $S$, two of which are isomorphic via an outer automorphism of $S$.

This is a toy example, but still it begins to suggest one approach to identifying a saturated fusion system $\mathcal{F}$: find a small collection of "nice" subsystems of $\mathcal{F}$ that generate $\mathcal{F}$, and show the corresponding amalgam of fusion systems is determined up to isomorphism by some suitable list of properties.

## Factor systems

Given a group $G$, the homomorphic images of $G$ are the factor groups $G/H$ for $H \trianglelefteq G$, so such images are parameterized by the normal subgroups of $G$. The morphic images of a fusion system $\mathcal{F}$ are parameterized by the strongly closed subgroups of $S$.

A subgroup $T$ of $S$ is *strongly closed* in $S$ with respect to $\mathcal{F}$ if for each $t \in T$, we have $t^{\mathcal{F}} \subseteq T$. In Example 1.1, if $H \trianglelefteq G$ then $S \cap H$ is strongly closed in $S$ with respect to $\mathcal{F}_S(G)$.

Let $T$ be strongly closed in $S$. We can define a fusion system $\mathcal{F}/T$ on $S/T$ such that the natural map $s \mapsto sT$ is a surjective morphism of fusion systems from $\mathcal{F}$ onto $\mathcal{F}/T$. The construction is the only one that could possibly work, and it is easy to show it works when $T \trianglelefteq \mathcal{F}$; in the general case, some effort is required. If $\mathcal{F}$ is saturated, then so is $\mathcal{F}/T$.

In Example 1.1 if $H \trianglelefteq G$ and $T = S \cap H$ then $\mathcal{F}/T \cong \mathcal{F}_{SH/H}(G/H)$.

Later we will define the notion of a "normal subsystem" of a saturated fusion system. If $\mathcal{E} \trianglelefteq \mathcal{F}$ has Sylow group $T$ then $T$ is strongly closed and we can define the factor system $\mathcal{F}/\mathcal{E}$ to be $\mathcal{F}/T$.

## Finite simple groups

We now, for the moment, leave the topic of fusion systems, and consider instead the finite simple groups and their classification. Recall:

**Theorem 1.7 (Classification Theorem)** *Each finite simple group is isomorphic to one of the following:*

*(1) A group of prime order.*

*(2) An alternating group $A_n$, for some $n \geq 5$.*

*(3) A finite simple group of Lie type.*

*(4) One of 26 sporadic simple groups.*

I'll assume we are all familiar with the groups of prime order and the alternating groups. The groups of Lie type are linear groups, so each has an associated prime: the characteristic of the field of the defining vector space; call this prime the *characteristic* of the group. The sporadic groups live in a natural way in no known infinite family of simple groups.

Eventually we will want to consider the 2-fusion systems of the simple groups, and use our functor to get information about those systems, and about simple 2-fusion systems in general. But first I want to discuss part of the proof of the Classification Theorem. To begin we need a few concepts and the associated notation.

## The generalized Fitting subgroup

Let $G$ be a finite group. Define $G$ to be *quasisimple* if $G = [G, G]$ and $G/Z(G)$ is simple. The *components* of $G$ are its subnormal quasisimple subgroups, where subnormality is the transitive extension of the normality relation on subgroups of $G$.

Let $E(G)$ be the product of the components of $G$; it turns out that $E(G)$ is a central product of the components: that is distinct components commute elementwise. Let $F(G)$ be the largest normal nilpotent subgroup of $G$ and $F^*(G) = F(G)E(G)$; then $F^*(G)$ is the central product of $F(G)$ with $E(G)$. We call $F^*(G)$ the *generalized Fitting subgroup* of $G$.

It turns out that $C_G(F^*(G)) = Z(F^*(G))$, so $F^*(G)$ controls the structure of $G$ in the sense that the image of $G$ in $Aut(F^*(G))$ under the conjugation map is isomorphic to $G/Z(F^*(G))$. Thus we can retrieve $G$, with little loss of information, from its generalized Fitting subgroup.

See [1] for a detailed discussion of the generalized Fitting subgroup.

The generalized Fitting subgroup is one of the important basic notions in the local theory of finite groups; this will become evident after more discussion.

Define $O(G)$ to be the largest normal subgroup of $G$ of odd order; Gorenstein called $O(G)$ the *core* of $G$. The CFSG focuses on 2-local subgroups of $G$; the cores of 2-locals cause significant difficulties in the CFSG. One of the advantages of working with 2-fusion systems is that such difficulties vanish, since cores disappear when we apply our functor, as the following lemma suggests:

**Lemma 1.8** *Let $* : G \to G/O(G) = G^*$ be the natural homomorphism $* : g \mapsto gO(G) = g^*$ and $S \in Syl_2(G)$. Then $* : \mathcal{F}_S(G) \to \mathcal{F}_{S^*}(G^*)$ is an isomorphism.*

One consequence of Lemma 1.8 is that if $\mathcal{F}$ is the 2-fusion system of a finite group then $\mathcal{F}$ is the 2-fusion system of an infinite number of finite groups. Hence it would seem that when applying our functor from finite groups to fusion systems, we lose a lot of information. While this is true, it may also be true that the lost information only serves to confuse many issues, and it may be an advantage to discard it.

Let $L_0$ be the preimage in $G$ of $E(G/O(G))$ and $L(G) = L_0^\infty$ be the last term in the derived series for $L_0$. We call $L(G)$ the *layer* of $G$. Observe that $L_0 = L(G)O(G)$.

The following result is due to Gorenstein and Walter; see [1] for a proof, modulo an appeal to the Schreier Conjecture.

**Theorem 1.9 (L-Balance Theorem)** *For each 2-subgroup $P$ of $G$ we have $L(C_G(P)) \leq L(G)$.*

The groups of Lie type of characteristic 2 have a different 2-local structure than those of odd characteristic. We seek to capture that difference in the general finite group in abstract group theoretic terms, rather than in the context of linear groups.

Define $G$ to be of *component type* if $L(C_G(t)) \neq 1$ for some involution $t$ of $G$; roughly speaking, the centralizer in $G$ of some involution has a component. Define $G$ to be of *characteristic 2-type* if $F^*(H) = O_2(H)$ for each 2-local subgroup $H$ of $G$.

**Remark 1.10** If $G$ is a simple group of Lie type and even characteristic, then $G$ is of characteristic 2-type. On the other hand almost all simple groups of Lie type and odd characteristic, other than $L_2(q)$, are of component type. The alternating groups $A_n$ for $n > 8$ are of component type. Some sporadic groups are of component type and some are of characteristic 2-type.

In short, if we seek to partition the simple groups into “even” and “odd” groups in terms of their 2-local structure, and in such a way that the groups of Lie type and even characteristic are "even", while those of Lie type and odd characteristic are odd, then we are led to define the even groups to be those of characteristic 2-type and the odd groups to be those of component type. Later we will see that this odd-even partition of the simple groups is perhaps not the optimal choice. But first let us see that, at least generically, each simple group is either odd or even using this definition:

**Theorem 1.11 (Gorenstein-Walter Dichotomy Theorem)** *Assume $O(G) = 1$ and $m_2(G) > 2$. Then $G$ is of component type or characteristic 2-type.*

See [3] for a proof of the Dichotomy Theorem.

Here $m_2(G)$ is the 2-rank of $G$: the maximum $m$ such that $G$ contains a subgroup that is the direct product of $m$ groups of order 2. The groups of 2-rank 2 should be thought of as "small" groups. Thus the Dichotomy Theorem says that, generically, each core-free finite group is either odd or even. Then the proof of the CFSG treats the small simple groups, the odd simple groups, and the even simple groups, using different methods for each type of group.

We are interested in simplifying the treatment of the odd simple groups. The most obvious advantage gained by treating the odd simple groups (as odd fusion systems) in the category of 2-fusion systems, comes from avoiding obstructions presented by cores of 2-locals, since, by Lemma 1.8, these cores vanish when we apply our functor.

In the treatment of groups of component type, the biggest obstacle mounted by core obstruction arises from the necessity to verify the B-Conjecture:

**B-Conjecture.** If $O(G) = 1$ then for each involution $t$ in $G$, we have $L(C_G(t)) = E(C_G(t))$.

The proof of the B-Conjecture is difficult and indirect. See [3] for more discussion of the B-Conjecture.

Given a simple group $G$ of component type, and assuming the B-Conjecture, we can consider the set $\mathfrak{C}(G)$ of components of centralizers of involutions. If $L$ is a component of $C_G(t)$ for some involution $t$, $s$ is an involution centralizing $t$ and $L$, and $G_s = C_G(s)$, then $L$ is a component of $C_{G_s}(t)$, so by L-Balance and the B-Conjecture, we have $L \leq L(G_s) = E(G_s)$. Indeed there exists a component $K$ of $G_s$ such that either $K \neq K^t$ and $L = E(C_{KK^t}(t))$ is an image of $K$, or $L$ *pumps up* to $K$: $K = K^t$ and $L$ is a component of $C_K(t)$. Keeping track of the pump up "ordering" on $\mathfrak{C}(G)$ and playing some combinatorial games, we are able to pin down the centralizer of an involution possessing a "maximal" member of $\mathfrak{C}(G)$. Then, as in the Brauer program, we identify $G$ from this centralizer. I'll be a bit more precise about what such a centralizer looks like later.

We seek to make an analogous argument in the category of 2-fusion systems. To do so, we must translate notions like "simple", "quasisimple", "component", etc., and theorems like L-Balance and the Dichotomy Theorem into analogous results on 2-fusion systems. The first crucial step in that process is to identify a notion of "normal subsystem" of a saturated fusion system.

## Normal subsystems

Let $\mathcal{F}$ be a fusion system on a $p$-group $S$. We begin with the notion of an $\mathcal{F}$-invariant subsystem. There are at least three equivalent definitions of such a system; here is one. Let $\mathcal{E}$ be a subsystem of $\mathcal{F}$ on $T$. We say that $\mathcal{E}$ is *$\mathcal{F}$-invariant* if $T$ is strongly closed in $S$ with respect to $\mathcal{F}$ and for each $P \leq Q \leq T$, each $\phi \in \hom_{\mathcal{E}}(P,Q)$, and each $\alpha \in \hom_{\mathcal{F}}(Q,S)$, we have $\phi\alpha^* \in \hom_{\mathcal{E}}(P\alpha, T)$, where $\phi\alpha^* = \alpha^{-1}\phi\alpha$.

The notion of $\mathcal{F}$-invariance is well behaved, but it has one draw back: even when $\mathcal{F}$ is saturated, an invariant subsystem need not be saturated. Fortunately there is an easy way to correct this.

Assume $\mathcal{F}$ is saturated and define a subsystem $\mathcal{E}$ of $\mathcal{F}$ to be *weakly normal* in $\mathcal{F}$ if $\mathcal{E}$ is $\mathcal{F}$-invariant and saturated. Finally $\mathcal{E}$ is *normal* in $\mathcal{F}$ if $\mathcal{E}$ is weakly normal in $\mathcal{F}$ and satisfies the *extension condition*: for each $\alpha \in Aut_{\mathcal{E}}(T)$, $\alpha$ extends to $\hat{\alpha} \in Aut_{\mathcal{F}}(TC_S(T))$ such that $[\hat{\alpha}, C_S(T)] \leq Z(T)$. Write $\mathcal{E} \trianglelefteq \mathcal{F}$ to indicate that $\mathcal{E}$ is normal in $\mathcal{F}$.

If $P$ is a subgroup of $S$ normal in $\mathcal{F}$ then $\mathcal{F}_P(P) \trianglelefteq \mathcal{F}$. In Example 1.1, if $H \trianglelefteq G$ then $\mathcal{F}_{S\cap H}(H) \trianglelefteq \mathcal{F}_S(G)$. The converse is in general false; as we saw in an earlier example, if $G$ is simple and $S$ abelian then $S \trianglelefteq \mathcal{F}_S(G)$ but $S$ is not normal in $G$.

Define $\mathcal{F}$ to be *constrained* if there is a centric subgroup of $\mathcal{F}$ normal in $\mathcal{F}$. In Example 1.1, if $F^*(G) = O_p(G)$ then $\mathcal{F}_S(G)$ is constrained as $C_G(F^*(G)) = Z(F^*(G))$. Define a *model* of a constrained system $\mathcal{F}$ to be a group $G$ with $F^*(G) = O_p(G)$ and $\mathcal{F}_S(G) = \mathcal{F}$. The topologists have shown in [5] that:

**Theorem 1.12 (Model Theorem)** *If $\mathcal{F}$ is a constrained saturated fusion system then $\mathcal{F}$ has a model $G$, and $G$ is unique up to an isomorphism which is the identity on $S$.*

**Theorem 1.13** *Let $\mathcal{F}$ be a constrained saturated fusion system with model $G$. Then the map $H \mapsto \mathcal{F}_{S\cap H}(H)$ is a bijection between the normal subgroups of $G$ and the normal subsystems of $\mathcal{F}$.*

The invariance condition is part of the definition of "normal subsystem" to insure our functor is bijective in Theorem 1.13.

Given a notion of "normal subsystem", we can now translate many notions from finite group theory to analogous notions about saturated fusion systems. As in the case of groups, subnormality for fusion systems is the transitive extension of the normality relation. Our saturated system $\mathcal{F}$ is *simple* if it has no nontrivial normal subsystem.

There is a smallest normal subsystem $\mathcal{E}$ of $\mathcal{F}$ such that $\mathcal{F}/\mathcal{E}$ is the system of a $p$-group; denote this system by $O^p(\mathcal{F})$. Define $\mathcal{F}$ to be *quasisimple* if $\mathcal{F} = O^p(\mathcal{F})$ and $\mathcal{F}/Z(\mathcal{F})$ is simple. Define the *components* of $\mathcal{F}$ to be its subnormal quasisimple subsystems.

It can be shown that $\mathcal{F}$ has a normal subsystem $E(\mathcal{F})$ that is the central product of the components of $\mathcal{F}$. Further $E(\mathcal{F})$ centralizes $O_p(\mathcal{F})$ and $\mathcal{F}$ has a normal subsystem $F^*(\mathcal{F})$ which is a central product of $O_p(\mathcal{F})$ and $E(\mathcal{F})$. Call $F^*(\mathcal{F})$ the *generalized Fitting subsystem* of $\mathcal{F}$; it can be shown that $C_{\mathcal{F}}(F^*(\mathcal{F})) = Z(F^*(\mathcal{F}))$.

**Theorem 1.14 (E-Balance Theorem)** *For each $P \in \mathcal{F}^f$, $E(C_{\mathcal{F}}(P)) \leq E(\mathcal{F})$.*

Define $\mathcal{F}$ to be of *characteristic p-type* if for each $1 \neq P \in \mathcal{F}^f$, we have $N_{\mathcal{F}}(P)$ is constrained. Define $\mathcal{F}$ to be of *component type* if for some $P \in \mathcal{F}^f$ of order $p$, $E(C_{\mathcal{F}}(P)) \neq 1$.

**Theorem 1.15 (Dichotomy Theorem for Fusion Systems)** *Let $\mathcal{F}$ be a saturated fusion system on a p-group $S$. Then $\mathcal{F}$ is either of characteristic p-type or of component type.*

The Dichotomy Theorem for fusion systems is stronger and has a more elegant statement than the Dichotomy Theorem for groups. It is also easier to prove.

## Beginning the program

Given the Dichotomy Theorem for Fusion Systems it makes sense to attempt to classify the simple 2-fusion systems of component type using the classification of the simple groups of component type as a template. In actual fact I propose to do something a bit different, but a discussion of those changes is perhaps best put off for a while.

The CFSG proceeds by induction on the group order, so one considers a simple group of minimal order subject to not being on the list of "known" simple groups. In such a group $G$ each proper simple section of $G$ is known. We will make a related assumption on our fusion systems.

Let $\mathcal{K}$ be the class of "known" simple 2-fusion systems, and $\tilde{\mathcal{K}}$ the class of "known" quasimple 2-fusion systems: those whose central factor system is in $\mathcal{K}$. I'll say a few words about these two classes shortly.

Let $\mathcal{F}$ be a saturated fusion system on a 2-group $S$. Define $\mathfrak{C}(\mathcal{F})$ to be the set of *components of centralizers of involutions* in $\mathcal{F}$; that is $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ if there exists some involution $t$ in $S$ and a conjugate $(\bar{t}, \bar{\mathcal{C}})$ of $(t, \mathcal{C})$ such that $\bar{t}$ is fully centralized and $\bar{\mathcal{C}}$ is a component of $C_{\mathcal{F}}(\bar{t})$. Thus $\mathcal{F}$ is of component type if $\mathfrak{C}(\mathcal{F})$ is nonempty. We will assume that each member of $\mathfrak{C}(\mathcal{F})$ is in $\tilde{\mathcal{K}}$.

Notice that we must pass to a conjugate $(\bar{t}, \bar{\mathcal{C}})$ with $\bar{t}$ fully centralized, so that we can apply Lemma 1.2 to insure that $C_{\mathcal{F}}(\bar{t})$ is saturated. This is necessary as components are only defined for saturated systems.

As in the CFSG, we have the pump up relation on $\mathfrak{C}(\mathcal{F})$, and we wish to show that if $\mathcal{C}$ is "maximal" with respect to this relation then the centralizers of involutions centralizing $\mathcal{C}$ are controlled. Finally we want to show the existence of such centralizers forces $\mathcal{F}$ to be isomorphic to a member of $\mathcal{K}$. Let us see in more detail what this means for groups:

Let $G$ be a finite group with $O(G) = 1$, satisfying the B-conjecture. Let $L \in \mathfrak{C}(G)$ have no proper pumpups, and set $K = C_G(L)$. Then (essentially) either

(1) $L \in \mathrm{Comp}(G)$, or

(2) $L$ is *standard* in $G$: that is $N_G(L) = N_G(K)$, $L$ commutes with none of its conjugates, and $K$ is tightly embedded in $G$.

Here a subgroup $K$ of $G$ is *tightly embedded* in $G$ if $|K|$ is even, but $|K \cap K^g|$ is odd for all $K \neq K^g$. It can be shown that in case (2), a Sylow 2-subgroup $Q$ of $K$ is small; namely either $m_2(K) = 1$ or $Q$ is elementary abelian, and then in the latter case, even $Q \cong E_4$. Thus the centralizer $C_G(t)$ of an involution $t$ in $K$ closely resembles the centralizer $C_{\bar{G}}(\bar{t})$ of some involution $\bar{t}$ in some known simple group $\bar{G}$, and this can be used to show $G \cong \bar{G}$ is known.

For example if $\bar{G} \cong A_n$ with $n > 8$ then $\bar{G}$ has a standard subgroup $\bar{L} \cong A_{n-4}$ with $\bar{Q} \cong E_4$. And if $\bar{G}$ is of Lie type over a field of odd order $q$, then usually $\bar{G}$ has a standard subgroup $\bar{L}$ of Lie type over $\mathbf{F}_q$ with $\bar{K} \cong SL_2(q)$, so that $\bar{Q}$ is quaternion.

As a first step toward proving $K$ is tightly embedded in $G$, one uses the condition that $L$ has no proper pumpups to show $L \in \mathrm{Comp}(C_G(i))$ for each involution $i \in K$. With a little care, it is possible to establish an analogous statement for fusion systems. One can also define the notion of a "tightly embedded subsystem" of a saturated fusion system, and prove the necessary theorems for such subsystems. But then we encounter a difficulty:

**Problem.** If $\mathcal{F}$ is a saturated fusion system and $\mathcal{E}$ is a subsystem of $\mathcal{F}$, we do not know how to define the normalizer or centralizer in $\mathcal{F}$ of $\mathcal{E}$, except in very special situations.

Because of the Problem, it is not straightforward to define a notion of a "standard subsystem" of a fusion system analogous to the notion of a standard subgroup defined above, but it is possible.

In short, the necessary notions from the CFSG do not all translate to fusion systems in a straightforward manner, but by and large it seems that such difficulties can be overcome. We will return to such details later; first let us discuss $\mathcal{K}$.

## The class $\mathcal{K}$ of known simple 2-fusion systems

Let $\mathcal{F}$ be a saturated fusion system on a 2-group $S$. Recall that $\mathcal{F}$ is *exotic* if $\mathcal{F}$ is realized by no finite group. There is one known class of exotic simple 2-fusion systems: the exotic *Benson-Solomon systems* $\mathcal{F}_{Sol}(q)$, for $q$ an odd prime power. If $\mathcal{F}$ is such a system then $\mathcal{F}$ has one class of involutions $z^{\mathcal{F}}$ and $C_{\mathcal{F}}(z)$ is the 2-fusion system of $Spin_7(q)$, which is quasisimple, so $\mathcal{F}$ is of component type. The isomorphism type depends only on the 2-share $(q^2-1)_2$ of $q^2-1$, not on $q$. The systems were "discovered" by Benson in a topological context, and earlier by Solomon as part of the CFSG, but these "discoveries" took place before the notion of a fusion system really existed.

So assume $\mathcal{F}$ is simple and realized by a finite group $G$. Then it is easy to see that we may choose $G$ to be simple, so we need to examine the known simple groups $G$ to see when $\mathcal{F}_S(G)$ is simple. A sufficient condition is to show that, first, $S$ is the smallest nontrivial strongly closed subgroup of $S$, and, second, that