1 Introduction: Three Faces of Intuitionism

Intuitionism, a revisionary movement in the foundations of mathematics, holds that mathematics and its objects must be humanly graspable. L. E. J. Brouwer founded it in his 1907 dissertation. In the 112 intervening years it developed a subtle and innovative mathematical face, established by Brouwer himself; a formal logical face, decried by Brouwer, but championed by his students and grand-students; and a philosophical face, initiated by Brouwer's student Arend Heyting and expanded by Michael Dummett.

This Element has pedagogical and philosophical goals. Pedagogically, I want to show you enough of the mathematical and logical aspects of intuitionism for you to see how subtle and interesting they are, and to enable you explore these further. Philosophically, I'll sketch a systematic philosophical grounding for intuitionistic mathematical thought, the 'intuitionistic standpoint'. I'll derive it from Brouwer's work, but – *pace* Brouwer – I'll use formal intuitionism to make it precise. Current philosophical foundations for intuitionism are, I shall say, at best a partial picture of this standpoint. I'll use intuitionism's mathematical and logical faces to give a fuller picture.

1.1 The Mathematical Face of Intuitionism

The early twentieth century was a turbulent time for mathematics. On the one hand the mathematical community was busy integrating the nineteenth century's revolutionary changes: non-Euclidean geometries (introducing abstract spaces and structures lacking physical avatars), the new algebra (viewing those abstract structures as mathematical objects in their own right), and Cantor's set theory (creating a consistent theory of infinite objects). This last was perhaps the most sweeping. It closed the ancient rift between the discrete and the continuous and it provided a common foundation for all of mathematics.

On the other hand, that very success raised problems. Mathematically, set theory engendered paradoxes – Russell's paradox is most famous –that tainted everything built upon it. So in the first third of the twentieth century, mathematicians worked mightily to rebuild set theory in a way that avoided those paradoxes. But philosophically, even before the paradoxes, there were objections to the infinitary set theoretic way of thinking. Cantor's continuum, for instance, is four layers deep in infinity; and set theory introduced higher and higher levels of infinity. Finite human minds, the objectors declared, cannot grasp such things.

So the twentieth century saw a spate of proposals designed to provide a finite, human grasp of mathematics while preserving the heart of modern mathematics. 'Hilbert's programme' was one of these. Formalise infinitary theories, he said: that is, axiomatise each theory in a regimented formal language and graft those axioms onto a system of logical axioms and rules expressed in the same formal language. Those systems were to be finitely graspable; and examining their proof structures would show them consistent and thus defuse paradoxes once and for all. This study of formal systems – Hilbert initially called it 'proof theory' and then 'metamathematics' – is the heart of contemporary mathematical logic, though it often loses Hilbert's 'finitary' flavour.

Gödel's incompleteness theorems scotched the technical side of Hilbert's original programme. But Brouwer objected to the programme on deeper grounds: Mathematical assertions, for Brouwer, are never empty formulae. He went directly to the offending mathematics, rebuilt it from the bottom up and strove to preserve our human grasp of its objects at every step. That is his 'intuitionistic mathematics', an innovative and rewarding enterprise. Brouwer developed it during his career, and to some extent his students and grand-students carried on the project.

Three things characterise this intuitionistic mathematics. First, its objects must be constructible; and its infinite objects – real numbers, functions over real numbers, the continuum itself – must be constructible in a way that makes them intuitively, finitely graspable. Second, it is alert from the very beginning to the fact that this demand for finite grasp brings with it a degree of indeterminacy: not only epistemic indeterminacy, but ontological indeterminacy as well. (There are things about which the world, reality, is simply undetermined.) And third, it deviates from standard (now called 'classical') set theory–based mathematics:[1] it refrains from asserting some common classical theorems; it makes subtle distinctions between classically equivalent notions; and it even proves theorems that are classically simply false. Brouwer's 'uniform continuity theorem' (every total function on a closed interval is uniformly continuous) is most famous.[2]

Brouwer accompanied his development of intuitionistic mathematics with unrelenting polemics against classical mathematics and against Hilbert's formalist programme. Classical mathematics, he said, assumes determinacy where there should be indeterminacy; introduces objects that in fact do not exist; and then builds castles in the air based on those nonentities. And, contra formalism, Brouwer rejected replacing intuitive thought with bare syntax, and he most stridently opposed the formalist use of logic. He had particular animus towards what has come to be called 'classical logic', the logic that Hilbert laid at the base of every formal system (most notably its 'law of the excluded middle').[3] That logic, he said, is a source for

[1] Brouwer [1908A] introduced the term 'classical mathematics' as the foil for intuitionism. Fraenkel [1923] picked up this usage and popularized it.

[2] I'll explain the terms in sub-sections 2.2.1 and 2.2.4.

[3] Hilbert [1923] laid a basis for the formal logical system. Its full codification in Hilbert and Ackermann [1928] is the *locus classicus*. The term "classical logic" as a foil for the logic of intuitionism goes back to Wavre [1926].

misplaced determinacy and existential excess. For Brouwer, purely logical manipulation must never replace actual intuitive construction.

1.2 The Logical Face

Given Brouwer's animus, one would hardly expect intuitionists to produce formal systems for 'intuitionistic logic' and for branches of intuitionistic mathematics. Yet that is exactly what happened. Heyting [1930] and [1930A] contain formal systems for intuitionistic logic and parts of intuitionistic mathematics, and from the 1960s various formal systems for the intuitionistic theory of real numbers and functions over real numbers appeared. All this despite – actually, I will suggest, because of – Brouwer's strident anti-formalism.

The 1920s and early 1930s witnessed internecine skirmishes between Brouwer and Hilbert.[4] The public issues were mathematical and logical: Hilbert opposed any deviation from classical mathematics or from classical logic; Brouwer opposed formalist foundations for mathematics. Hilbert won:

- He won mathematically: intuitionistic mathematics is barely practiced and hardly taught today.
- He won logically: classical logic is still the logic of choice, and excluded middle remains a powerful mathematical tool.
- He won metamathematically: the study of formal systems is a staple of modern foundational studies. Indeed, even Heyting's formal systems and the subsequent formalisations of intuitionistic analysis underwent their own metamathematical examinations, often using classical logic and applying classical tools.[5] Most modern students know only this about intuitionism.

1.3 Philosophy

Intuitionistic mathematics is rich and innovative and well worth studying on its own. But, in fact, it rests on a subtle philosophy, a philosophy ultimately abetted by formal intuitionism. Ironically, however, modern students seeking a philosophical ground for what they do know of intuitionism – intuitionistic logic – find Heyting's 'proof' interpretation for the logical particles in formal languages; and then they encounter Michael Dummett's extension of Heyting's conception to all of language in general. Nowadays intuitionistic philosophy is all about linguistic meaning and formal truth. How deeply un-Brouwerian!

[4] van Dalen [2005] pp. 599–643 details the conflict.
[5] Indeed, a Buffalo conference in 1968, shortly after Brouwer's death, and its proceedings called *Intuitionism and Proof Theory* (Kino, Myhill and Vesley [1970]) fully solidified the metamathematical study of intuitionistic systems.

I shall argue, however, that the philosophical ground of intuitionistic mathematics (and with it, intuitionistic logic) is a systematic union of phenomenological, epistemological and ontological doctrines, the 'intuitionistic standpoint'. I will extract these doctrines from the content and practice of intuitionistic mathematics, from the notions of finite grasp and indeterminacy that lie at its core. Moreover, I'll show that the formalised intuitionism actually clarifies those notions and even gives a platform to show the true semantic side to intuitionism, a side far subtler than Heyting's or Dummett's partial pictures.

1.4 Preview

Section 2 starts with brief sketches of the relevant classical mathematical notions, of Hilbert's programme and of Brouwer's objections. Then it introduces the core of intuitionistic mathematics from the natural numbers through the basic theory of real numbers and real valued functions (what we call 'real analysis'). It includes some detailed proofs exemplifying intuitionistic mathematical reasoning. (These can be skipped without losing the gist of the story.) Then it offers a careful account of the 'fan theorem', the main result from which the uniform continuity theorem follows. It shows that Brouwer pulled off an historic mathematical coup about the structure of the continuum.

Section 3 sketches two distinct programmes formalising aspects of intuitionism: first, Heyting's formal systems for intuitionistic logic and for intuitionistic number theory together with a taste of its metamathematics. Second, the central formal systems of intuitionistic analysis together with a main meta-theorem concerning these systems. It emphasises how these systems for analysis generalise the ideas underlying the proof of the fan theorem. It concludes with brief comparisons to two classical mixes of constructivity and analysis.

Building on all this, Section 4 broadly outlines the intuitionistic standpoint. I derive the aspects concerning experience and knowledge and those concerning objects from intuitionistic mathematics and from the main parts of formalised intuitionistic analysis. I will also use formal intuitionistic logic to present the more nuanced semantic view that emerges from Brouwer's thought. In the end I will trace the theme of 'finite grasp' through all aspects of intuitionism and point out the nuanced intuitionistic conception of indeterminacy – both epistemic and ontological indeterminacy.

To present this picture of mathematical intuitionism, I have selected core topics that give a coherent picture and prepare the reader for further study. A brief 'Afterword' lists important further topics.

This Element aims for readers with a background in logic and philosophy, and it assumes some familiarity with logical notions and notation. Here's a brief lexicon:

Expression	Meaning
$\boldsymbol{p \wedge q}$	p and q
$\boldsymbol{p \vee q}$	p or q
$\boldsymbol{p \rightarrow q}$	if p then q
$\boldsymbol{p \leftrightarrow q}$	$\boldsymbol{p}$ if and only if (iff)q
$\sim\boldsymbol{p}$	p is strongly false (a special meaning introduced in Section 2)
$\exists\, \boldsymbol{x\, Px}$	there is at least one x such that Px holds
$\forall\, \boldsymbol{x\, Px}$	every object x is such that Px holds of it[6]
$\{\boldsymbol{x}_1, \ldots, \boldsymbol{x_n}, \ldots\}$	the collection of objects $x_1, \ldots, x_n, \ldots$[7]
$\varnothing$	the empty set
$\{\boldsymbol{o} \mid \boldsymbol{A}(\boldsymbol{o})\}$	the collection of all objects o such that $A(o)$ holds
$\{\boldsymbol{x_n}\}_{\boldsymbol{n}}$	the sequence $x_1, x_2, \ldots$
$\vdash_S \boldsymbol{A}$	A is derivable in formal system S
$\boldsymbol{\Gamma} \vdash_S \boldsymbol{A}$	A is derivable from premise set Γ in formal system S

- $n, m, k, n_1, \ldots, m_1 \ldots, k_1, \ldots$ range over natural numbers
- $a, b, a_1, \ldots, b_1, \ldots$ range over rational numbers
- $r, r_1, \ldots$ range over real numbers
- $\alpha, \beta, \gamma, \alpha_1 \ldots, \beta_1 \ldots, \gamma_1, \ldots$ range over sequences
- $f, g, h, f_1, \ldots, g_1, \ldots, h_1, \ldots$ range over functions

Section 2 uses these symbols informally. Section 3 uses them both informally and in formal languages. The context will make the usage clear.[8]

2 The Mathematical Face of Intuitionism

2.1 Classical Foil and Formalist Foe

I'll build the Cantorian continuum starting from the ***natural numbers*** (the set $\mathbb{N} = \{0, 1, 2, \ldots\}$). Then I'll mention the mathematical and conceptual problems that this raises, briefly show how Hilbert's programme addresses both, and describe Brouwer's grounds for rejecting this solution.

2.1.1 The Cantorian Continuum

Natural Numbers

We can distinguish natural numbers from one another and can do arithmetic with them. This already allows us to order them: We say $n <_{\mathbb{N}} m$ if

[6] These last three are often restricted to some particular domain of objects.
[7] Following Brouwer, I'll generally take sequences to begin with x_1.
[8] When speaking informally, I'll use $\Rightarrow$ and $\Leftrightarrow$ rather than $\rightarrow$ and $\leftrightarrow$.

$$\exists\, k_{\neq 0}(m + k) = n. \tag{2.1}$$

We assume that the principle of mathematical induction holds: for any property A of natural numbers if A holds of 0 and if A is preserved when we add 1, then A holds of all the natural numbers. Formally:

$$[A(0) \wedge \forall n(A(n) \rightarrow A(n + 1))] \rightarrow \forall n A(n). \tag{2.2}$$

This in turn allows us to prove such general features as commutativity of addition and multiplication

$$\forall n \forall m(n + m = m + n), \tag{2.3}$$

$$\forall n \forall m(n \cdot m = m \cdot n), \tag{2.4}$$

and the general least-number principle:

$$\exists x A(x) \rightarrow \exists x(A(x) \wedge \forall k_{<x} \sim A(k)). \tag{2.5}$$

Integers and Rationals

$\mathbb{N}$ extends naturally to the set $\mathbb{Z}$ of the ***integers*** (adding the negative numbers to $\mathbb{N}$) with its usual arithmetic and ordering. And that in turn extends naturally to the fractions: pairs $\frac{n}{d}$ where n and d are integers, and $d \neq 0$. Given fractions $\frac{n}{d}$ and $\frac{n'}{d'}$, we say that $\frac{n}{d} < \frac{n'}{d'}$ when $n \cdot d' <_{\mathbb{Z}} n' \cdot d$; and that $\frac{n}{d} = \frac{n'}{d'}$ when $n \cdot d' =_{\mathbb{Z}} n' \cdot d$. That equality is an ***equivalence relation*** – it is reflexive, symmetric and transitive – so it produces disjoint ***equivalence classes***.

A ***rational number*** is one of these equivalence classes; $\mathbb{Q}$ is the set of all rational numbers. Arithmetic operations, identity $r_1 =_{\mathbb{Q}} r_2$, and order $r_1 <_{\mathbb{Q}} r_2$ on rational numbers work with representative fractions. These operations and relations are independent of the particular fractions we use in order to represent the rational numbers.

$\mathbb{R}$ and its Fine Structure

A sequence $\alpha = \{a_n\}_n$ of rational numbers is ***convergent*** when its elements get closer and closer to one another: as the sequence progresses, the distance between elements gets smaller than any given small number. For convenience we take the small numbers to be reciprocals of powers of 2. So formally this comes to

$$\forall k_{\in \mathbb{N}}\, \exists\, n_{\in \mathbb{N}} \forall m_{\in \mathbb{N}}(|a_n - a_{n+m}| <_{\mathbb{Q}} 2^{-k}). \tag{2.6}$$

($|c|$ is the absolute value of c.[9]) Two such sequences, $\alpha = \{a_n\}_n$ and $\beta = \{b_n\}_n$, ***coincide*** (we write $\alpha \simeq \beta$) when *their* respective elements get closer and closer:

$$\forall k \exists n \forall m (|a_{n+m} - b_{n+m}| <_{\mathbb{Q}} 2^{-k}). \tag{2.7}$$

Thus, for example, the familiar decimal expansions are just sequences, $\{x_n \cdot 10^{-n}\}_n$, where every x is an integer, and which satisfy

$$|x_n \cdot 10^{-n} - x_{n+1} \cdot 10^{-(n+1)}| \leq_{\mathbb{Q}} 10^{-(n+1)}. \tag{2.8}$$

A ***real number***, r, is an equivalence class of coincident convergent sequences. We define

$$r_\alpha =_{\text{df}} \{\beta | \alpha \simeq \beta\} \tag{2.9}$$

and say r_α is **generated by** α.

Finally, $\mathbb{R}$ is the set of real numbers. This is the one-dimensional continuum, the 'real line'.

We do arithmetic on real numbers by working with their generating sequences. So if $\alpha = \{a_n\}_n$ and $\beta = \{b_n\}_n$ then

$$r_\alpha + r_\beta = \{a_n + b_n\}_n \tag{2.10}$$

and similarly for the other arithmetic operations. Once again, these operations are independent of the sequences representing the real numbers.

Order

The real numbers are ordered in a natural way, and this too is expressed via representing sequences. If $\alpha = \{a_n\}_n$ and $\beta = \{b_n\}_n$, then ***r_α is less than r_β*** (we write $r_\alpha <_{\mathbb{R}} r_\beta$) if

$$\exists k_{\in \mathbb{N}} \exists n_{\in \mathbb{N}} \forall m_{\in \mathbb{N}} ((b_{n+m} - a_{n+m}) >_{\mathbb{Q}} 2^{-k}). \tag{2.11}$$

This too is representative-independent.

Topology

The study of 'nearness' belongs to topology. While it may be done very abstractly, here, on the real line, we can use the natural ordering $<_{\mathbb{R}}$ to define intervals, and use those to express a sort of proximity. If $r_1 <_{\mathbb{R}} r_2$, the ***open interval*** (r_1, r_2) is

[9] $|c|$ is c if $c \geq 0$, and $|c|$ is $-c$ if $c < 0$.

$$\{r|r_1 <_{\mathbb{R}} r <_{\mathbb{R}} r_2\}. \tag{2.12}$$

This is the set of real numbers ***between*** r_1 and r_2.[10] The ***closed interval*** $[\boldsymbol{r}_1, \boldsymbol{r}_2]$ is (r_1, r_2) plus its endpoints. That is:

$$\{r|r_1 \leq_{\mathbb{R}} r \leq_{\mathbb{R}} r_2\}. \tag{2.13}$$

$\mathbb{Q}$ and $\mathbb{R}$ both have the property that between any two distinct elements of either one of those sets lies a third element from the same set. This is called ***everywhere density.***

By contrast, $\mathbb{Q}$ and $\mathbb{R}$ differ regarding what's called the ***least upper bound (LUB)*** property. $\mathbb{R}$ has the property: If $S(\subseteq \mathbb{R})$ is bounded from above, that is, if

$$\exists x \forall y_{\in S}(y \leq_{\mathbb{R}} x), \tag{2.14}$$

then there is a real number r such that every open interval containing r also contains an element of S. But $\mathbb{Q}$, as we have defined it, does not have this property. $S = \{a_{\in \mathbb{Q}}|a^2 < 2\}$ is bounded above (by *e.g.* the rational number $\frac{3}{2}$), but S has no LUB in $\mathbb{Q}$.

Functions

If you think of a function as a correlation associating a unique value to each of its arguments, then a ***real valued function*** f correlates real numbers to real numbers. If the arguments come from a particular subset, $A(\subseteq \mathbb{R})$, A is called the domain of f.

A function f is ***continuous at a point x*** in its domain, if whenever an argument y in the domain is close to x, the value $f(y)$ will be close to $f(x)$. More precisely, for any open interval around $f(x)$ that we pick, we can find an open interval around x, such that for any y within that interval around x, $f(y)$ will be in the chosen interval around $f(x)$. Formally,

$$\forall k_{\in \mathbb{N}} \forall y_{\in A} \exists n_{\in \mathbb{N}}(|x - y| <_{\mathbb{R}} 2^{-n} \rightarrow |f(x) - f(y)| <_{\mathbb{R}} 2^{-k}). \tag{2.15}$$

We say that ***f is continuous on A*** if it is continuous at each $x \in A$.

The ***Intermediate Value Theorem*** says that if f is continuous on $[r_1, r_2]$, then every real number between $f(r_1)$ and $f(r_2)$ is the value of the function for some argument in $[r_1, r_2]$. This captures the pictorial idea that a continuous function has no gaps.

[10] One can say that at every point in an open interval there is 'room to move around' in either direction without leaving the interval.

A special case is **Bolzano's Theorem**: If f is continuous on $[r_1, r_2]$, $f(r_1) > 0$ and $f(r_2) < 0$, then there is an $r_3 \in (r_1, r_2)$ such that $f(r_3) = 0$ (r_3 will be the LUB of the set $\{r | r \in (r_1, r_2) \wedge f(r) <_{\mathbb{R}} 0\}$).

The real valued function f is ***uniformly continuous*** on $A (\subseteq \mathbb{R})$, if

$$\forall k \exists n \forall x_{\in A} \forall y_{\in A} [(|x - y| <_{\mathbb{R}} 2^{-n}) \rightarrow (|f(x) - f(y)| <_{\mathbb{R}} 2^{-k})]. \tag{2.16}$$

Here, unlike (2.15), the requisite n depends only on k and is independent of the argument x. Thus, for instance, $f(x) = x^2$ is continuous on $\mathbb{R}$, but is not uniformly continuous on $\mathbb{R}$. It grows too fast: pick any k, then for any n we can always find a pair x, y in $\mathbb{R}$, such that $|x - y| <_{\mathbb{R}} 2^{-n}$ but $|x^2 - y^2| >_{\mathbb{R}} 2^{-k}$. On the other hand, if we take any closed interval, $[a, b]$, then $f(x) = x^2$ *is* uniformly continuous on that interval, because it is limited in how fast it can grow within the interval.

A central classical theorem is the **Classical Uniform Continuity Theorem**: If f is continuous on a closed interval, then f is uniformly continuous on that interval.

2.1.2 Consequences

Set theory provided a universal ontology for all of mathematics, and it introduced a new universe of heretofore unthought-of mathematical objects.

Universal Ontology

Not only are $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ sets; not only are rational and real numbers sets; but so too are the natural numbers, and, with them, the integers. We can, for instance, let $0 = \varnothing, 1 = \{\varnothing\}, \ldots, n = \{n - 1\}$, and then define arithmetic operations and order accordingly.

Similarly, pairs $\{x, y\}$, ordered pairs $\langle x, y \rangle$ and, in general, ordered n-tuples $x_1, x_2, \ldots$ are sets. So too are relations. For instance: $<_{\mathbb{R}}$ is the set

$$\left\{ < r_\alpha, r_\beta > | \alpha = \{a_n\}_n \wedge \beta = \{b_n\}_n \wedge \exists k_{\in \mathbb{N}} \exists n_{\in \mathbb{N}} \forall m_{\in \mathbb{N}} ((b_{n+m} - a_{n_m}) > 2^{-k}) \right\}. \tag{2.17}$$

Analogously, each of the order relations among elements of $\mathbb{N}$, of $\mathbb{Z}$ and of $\mathbb{Q}$ is a set. Moreover, a function, f, is a set too, $f = \{\langle x, y \rangle | f(x) = y\}$. Continuity is just a property of particular sets of this form. So it too is a set, the set of functions having this property. Everything in mathematics – every object, every property, every relation that we can define – is a set. And since sets are distinguished by their elements and only by their elements, the only actual relation we have is '$\in$'.

Large Sets

Set theory gives a precise account of ‘size’ (cardinality), and this brings with it a new universe of large sets.

Two sets, A and B, are ***equinumerous*** ($A \approx B$) if they are in $1 - 1$ correspondence; that is,

$$\exists f[\forall x_{\in A}\exists y_{\in B}(f(x) = y) \wedge \forall y_{\in B}\exists x_{\in A}(f(x) = y) \wedge \forall x\forall y(f(x) = f(y) \rightarrow x = y)]. \tag{2.18}$$

If $N \approx A$ we say that A is ***denumerably infinite*** (or ***denumerable***), and we call the correspondence f an ***enumeration*** of A.

$\mathbb{Q}$ is known to be denumerable, but Cantor showed that $\mathbb{R}$ is not. ($\mathbb{R}$ is non-denumerable, or, as we say, ‘uncountable’). Informally: Let $B(\subsetneq\mathbb{R})$ be the set of real numbers with decimal expansions consisting only of 0’s and 1’s.[11] Were B denumerable, we could line up the natural numbers in a column and match each with a decimal expansion of a real number in B, exhausting all of B. We could then go down the second column, changing the n th digit of the decimal expansion in the n th row (from 0 to 1 or 1 to 0). The result would be a decimal expansion (and hence a real number) in B that is not on the list. Since B is uncountable, so is $\mathbb{R}$.

We say that $\mathbb{N}$ and $\mathbb{Q}$ have the same ***cardinal number***, itself a certain set – we write $\|\mathbb{N}\| = \|\mathbb{Q}\|$ and say that Cantor proved this cardinal number to be smaller than the cardinal number of $\mathbb{R}$: $\|\mathbb{N}\| < \|\mathbb{R}\|$.

The process goes further. Given a set, A, define $\mathcal{P}(A)$, A’s ***power set***, as the set of all subsets of A.

$$\mathcal{P}(A) =_{\text{df}} \{x | x \subseteq A\}. \tag{2.19}$$

Cantor actually proved in general that $\|A\| < \|\mathcal{P}(A)\|$. This gives us a hierarchical universe of larger and larger sets.

2.1.3 Conceptual Problems

Cantor’s naïve picture is inconsistent. Russell’s set $R = \{x | x \notin x\}$ shows this, for both $R \in R$ and $R \notin R$ must hold. Mathematicians patched this by carefully designed axioms, ultimately Zermelo-Fraenkel set theory (ZF). But nonetheless two conceptual problems made this set theoretic picture untenable for Brouwer.

[11] Of course, these real numbers will have other representations as well.