1 What Are We Talking About?

There is an age-old question running through the philosophy of mathematics: are numbers (or any of the structures that mathematics studies) invented or discovered?[1] The primary focus of this Element will not be to cover or to contribute to this debate (at least, not directly) but rather to focus on the *question itself*, which, as we will see, is by no means as clear as it may at first seem.

We will look at a number of core concepts and issues, all of which need to be addressed in some way in order to even begin to understand what exactly we are talking about – what exactly we are asking – when we posit the question in the first place.

This discussion could range over a great deal – far more than can be addressed in an introductory Element such as this. Similarly, it could take many possible assumptions or positions as its starting point. For example, it could begin with the assumption that the question is fundamentally philosophical and so has not much to do with the actual practice of mathematics. Or, it could look at the question, in the first place, as primarily a question for linguistics. It could take the question to be ethical – about what we value and why; or it could situate the question as properly formal – perhaps one that should initially be asked within the realm of, say, logical and mathematical symbols. The list goes on. What we take as the initial range of the question, or where we begin when we begin to try to understand it, will affect what we take as the main concepts and issues with which it interacts.

I do not claim to have identified the only proper way to begin our investigation, but hope it will highlight its complexity and depth, and so the importance of examining it in the first place.

The initial focus here – what we might initially take the question to be about – will be on the status of that which provides the foundations of mathematics as it is practised, in particular as it is practised *as true.* An immediate point to note is that we cannot survey all practising mathematicians to determine just how many do indeed practise mathematics as though it is true; but it is safe to say that there is enough interest in the question among practitioners of both mathematics and the philosophy of mathematics, who accept that our usual inclination is to believe in the truth of such claims as $2 + 2 = 4$, to justify the focus on the truth of mathematics taken here.

To push the point, the sort of truth generally ascribed to mathematical statements is an especially trustworthy truth – scientific, logical, and

[1] For an introduction to this introductory question, see www.youtube.com/watch?v=ujvS2K06dg4&list=TLPQMTcwNjIwMjHV9PBJnA4ltA&index=4

mathematical truths are generally held in higher regard than, say, social or sociological truths.

This focus will further narrow to the historically important and continuing debate between two main positions on just what might constitute this truth, namely 'constructivism' and 'Platonism'. These positions within the philosophy of mathematics roughly line up with what has commonly been called, in other philosophical fields, 'anti-realism' and 'realism' respectively. In broad strokes, the former takes the truth of mathematics to be constituted by human activity, language, minds, and so on – that is to say, it takes the position that the truth of mathematics largely depends on us. The latter takes the truth of mathematics to be independent of humans – such that it is, as Michael Resnik puts it (swapping out the word 'logical' with the word 'mathematical'),

> [t]he doctrine that statements attributing *mathematical* [author's addition] . . . properties or relationships . . . are true or false independently of our holding them to be true, our psychology, our linguistic and inferential conventions, or other facts about human beings. (Boghossian and Peacocke, p. 334)

Both these characterisations have their problems, and both have been contested as to whether they are the best frameworks within which we can or should characterise the differences between the two positions. So the central focus here, within the focus on these two positions, will be on just how best to understand and articulate their differences.

An immediate impact of the broad-brush characterisation we've just offered is that each position so characterised comes with its own strengths and weaknesses regarding the core concerns of the philosophy of mathematics. For anti-realism, thought of as encompassing the idea that mathematical truth is constituted (or constructed) by humans somehow, there is an immediate problem regarding the strength of this truth – how, for instance, if we adopt this position, can we account for the notion that the truth of mathematical statements is especially trustworthy and somehow different to other apparently human-constructed or human-dependent truths?

On the other hand, mathematical realism, thought of as encompassing the idea that mathematical truth is broadly independent of humans, encounters an immediate problem of access – for instance, how, if we adopt this position, can we know that we've managed to discover the 'real' mathematical truths, as opposed to, say, those we find most useful to us or perhaps to a small set determined by our own intellectual, biological functioning? Worse, how can we know if we're even close to discovering independent truth in an abstract field such as mathematics? If we've no way of accessing truth via empirical data, tests or our basic senses, how can that truth be confirmed or trusted?

But each position thus characterised also has its own particular advantages. Mathematical realism has the advantage of explaining, or at least articulating, the sort of strong, trustworthy truth generally attributed to mathematical claims. And, mathematical anti-realism has the advantage of explaining, or at least articulating the way we access mathematical truths – after all, if that truth is in essence our creation, all we need do is study relevant features of our own human biology, mind, and so on to see how we arrive at what we've come to believe as true in mathematics.

To better see why the relationship between these two positions is an important question within the philosophy of mathematics, note that while much has been written on the impacts of taking one or the other of realism or anti-realism regarding the truth of mathematics, there is a clear way in which the two positions themselves are, on closer inspection, quite difficult to distinguish from one another.

And so, our initial way into understanding the question itself runs quite quickly up against a thorny problem: what do you, the constructivist (or, broadly put, anti-realist), really mean when you ask whether mathematics is invented or discovered, and what do you, the realist, really mean when you ask the same thing? And how can we tell when we've understood each position as it would understand itself?

Of course, it is important to note that I intend 'constructivism' (which I take as broadly as possible here: i.e. as an instance of 'anti-realism' at large) and 'realism' to be cover terms for an immense array of particular positions within each camp, many of which are, again, quite different from each other. And some philosophers of mathematics would take this broad approach to be a mistake at the outset.[2] But the focus of this Element is not on separating the myriad specific positions in the literature, but on what happens at the very beginning – on how any one enquirer can (or cannot) understand the question of inquiry itself, and so whether or not 'the' question can make any sense in the first place.

Section 2 will focus on this conundrum – highlighting the extent to which the two (broad-brushed) positions may in fact be unable to be distinguished, or to be distinguished along the lines to which proponents on either side would agree. This is a significant problem: without agreed-on terms for the debate, the debate is stopped before it begins.

Stewart Shapiro notes this (1997), arguing that non-revisionist, anti-realist programs (i.e. programs that seek to retain and explain most of mathematics as it stands) are inter-translatable with classical logic, or traditional mathematics complete with its apparent realist commitments to numbers, sets and other

[2] Thanks to an anonymous referee for highlighting this point.

mathematical things. That's actually the way that anti-revisionist, anti-realist programs work in the first place – by translating apparently problematic number (etc.) talk into talk of other, apparently less problematic stuff. But, as we'll shortly explore in more depth, this translation works both ways: so anytime I like, I can equate your talk of mental constructions with my talk of numbers and vice versa.

Shapiro then argues that the problems associated with each program are inter-translatable also. So constructivism, for instance, ends up saddled with the same problems regarding epistemological access as traditional realism. This is all due to the fact that each non-revisionist, anti-realist program has to posit abstracts or use the notion of 'possibility' in a semi-concrete sense. But, the reference to semi-concrete abstract notions is as vexing a problem for the anti-realist as it is for the realist. That is, there is as much an access problem for anti-realism as for realism, just insofar as anti-realism inter-translates with traditional mathematics.

On the basis (mentioned earlier) that inter-translation works both ways, the problems each position encounters run in the opposite direction also: that is, so far as realism is inter-translatable with anti-realism, there is a burden on the mathematical realist to show how their posited reality differs from that of the anti-realist.

Ultimately, this Element argues that an effective defence of just such a difference needs a commitment to the notion of 'independence' regarding mathematical reality (or, at least a commitment to agree on what this might mean), which in itself involves a commitment to what we'll call the 'ontological' access problem *as a genuine problem* – essentially, this is the problem of how knowable mathematical truths are identifiable with a reality independent of us as knowers.

Section 3 outlines two ways in which we might understand what such 'access' might mean and explores some of the ways in which 'ontological' access has been conflated with what should, I suggest, be identified as something else entirely, namely 'epistemological' access. Ontology and Epistemology are two well established fields of study within the philosophical tradition. Broadly, the former deals with what is, and the latter with knowledge. The hope is that mapping the boundaries and potentially shared terms of Ontology and Epistemology along the potential division of constructivism and realism helps tease out what we may really be debating when we ask the question under investigation.

Section 3 specifically argues that if the ontological problem (recall, this is identified here as the problem of how knowable mathematical truths are identifiable with a reality independent of us as knowers) is defused, side-stepped or conflated with the 'epistemological' problem – that is, the problem of how *we come to know* mathematical truths – then it appears that nothing is gained by the realist notion of an independent reality and, in effect, nothing then distinguishes realism from anti-realism (or constructivism) in mathematics. That is, if the epistemological problem is the only playing ground for skirmishes between realism and anti-realism in mathematics, then the problem of effectively distinguishing the two positions remains.

That problem is not small. It boils down to this: that the 'what-is-perceived' and 'what-existed-prior-to-that-perception' must (initially at least) be distinguishable from each other, then shown to be (at least reliably to be believed as) one and the same when we 'get it right'. This is the ontological access problem for traditional (physical) realism and is precisely why such realism is the sceptic's favourite target. In the realist story, it is in the separability of independent reality (or, in Fregean terms, 'the referent') from our constructions or objects of belief, or our experiences of the 'mode of presentation', that the strong sense of objectivity and justification reside. The corresponding access problem is about overcoming this separateness while nonetheless retaining what is gained by that separateness itself.

Section 4 then examines how this all might help us agree on a notion of 'independence' able to effectively distinguish between the two sides of the debate and so overcome the twin problems of their talking past one another and their inter-translatability.

Section 5 argues that the notion of an independent mathematical reality will meet similar inter-translatability problems as covered in earlier sections, unless it does some sort of extra (or differentiable) work. This section argues that this differentiable work needs an appeal to a certain notion of what it is to be justified in mathematics.

The terrain here is very broad indeed, and is intended to be, as the Element is a discussion of the very broad strokes in which we might situate various positions regarding the foundations of mathematics, or better still with which each such position may be distinguished along our very general understanding of the initial question.

2 Inter-translatability

Looking at the 'inter-translatability' phenomenon in more detail, we return to what Shapiro argued are the best of the anti-realist programs – that is, those that seek to explain most of mathematics as it stands. To flesh out the argument a little more, note again that Shapiro argues that such programs involve at least as many of the epistemic and semantic problems as are involved in realism:

> In a sense the problems are equivalent – for example, a common maneuver today is to introduce a 'primitive' such as a modal operator, in order to reduce ontology. The proposal is to trade ontology for ideology. However in the context at hand – mathematics – the ideology introduces epistemic problems quite in line with the problems with realism. The epistemic difficulties with realism are generated by the richness of mathematics itself. (p. 5)

Recall that the specific sort of realism that Shapiro (1997) defends is holistic, which is why he does not address the ontological access problem from first principles, or 'from without' as it were.

Rather, his defence of his ontology of structures is via an argument extending Wright's notion of entitlement to the successor principle and thence to finite cardinality structure and the natural number structure:

> Wright argues that 'Entitlement of cognitive project does not . . . extend to matters of ontology'. At least in the case at hand, basic arithmetic, I would say that it does. *S* [a generalized person interested in the nature of mathematics] has reason to believe in the existence of the (*ante rem*) finite cardinality structures, and also in the existence of the natural number structure. The entitlement to the successor principle is a key element in the justification for the existence claim. (Shapiro, 2011, 146)

Wright's notion of entitlement is a general one – very loosely put, it grants justification to a belief just so long as denying that belief would be irrational or harmful to our overall life and our attempt to understand the world we encounter.[3] Shapiro (2011) gives a similarly holistic argument in which it is argued that our ability to coherently discuss a mathematical structure is evidence that it exists (p. 147).

In line with the equivalence argument presented at the beginning of this section, this Element suggests that the difficulties generated by equivalence itself and in turn by the sort of holistic realism that the equivalence argument supports, are arguably greater than the difficulties generated by a more traditional Platonism. Granted – the burden on the latter is both ontological and epistemological. The burden on the former, however, is to show how the ontology/ideology trade-off works in the realist's favour, specifically how it differentiates realism from anti-realism, or minimally, if perhaps an independent reality is *stipulated*, how that stipulation (given equivalence) avoids the ontological access problem. On the other hand, insofar as the ontological problem can be set aside, the burden on the holistic realist is then to show how (given equivalence) the mathematical reality posited is any different from the ideological universe of mathematical objects in anti-realism.

[3] For more on this, see Wright (2004).

That is, Shapiro has shown that denying the existence of mathematical objects requires at least as much philosophical justification as asserting it. But to this we can add that asserting the holistic ante rem existence of mathematical structures similarly requires just as much philosophical justification as asserting a more traditional Platonist position (i.e. both do, just insofar as they stipulate the independence of those structures, encounter the ontological access problem), but that the latter has the added advantage of being more than an ideology/ontology swap. That is, the latter retains an emphasis on the principal means by which realism may be differentiated from anti-realism – the concept of independence as a live, continuing issue, one that is unresolvable via inter-translatability arguments. So long as the cat on the mat is taken to independently exist, the best explanation of our corresponding belief that the cat is on the mat is provided by an account in which the ontological access problem remains acute: that is, one in which there is an inherent gap between what we know, how we know, and what, independently, is so.

The argument here, then, in some more detail, begins where Shapiro seeks to establish that an anti- realist who proposes to avoid a commitment to the existence of abstract objects such as numbers, sets or structures, is ultimately no better off – particularly on the epistemic front – than a realist who embraces the notion that such objects actually exist.

Insofar as it challenges the anti-realist notion that mathematics is about no more than linguistic or logical notions we are already familiar with, this argument is a powerful tool for the realist. But just so far as it challenges these anti-realist notions it jeopardises the realist's own proposed notion that mathematics is about an actual rather than a possible reality, especially any kind of non-derivative, independent realm.

The argument that Shapiro puts forward begins by showing that the anti-realist programs he examines and the realist's commitment to real abstract objects are, in fact, equivalent. That is, he shows that 'any insight that modalists [the particular group of anti-realists to whom Shapiro directs his argument] claim for their system can be immediately appropriated by realists and vice versa.

Moreover, the epistemological problems get "translated" as well' (p. 219).1

Very basically rendered, the argument compares a possible mathematical realm (one built from a plural quantifier, a contractibility quantifier, or an operator for logical possibility) with a real mathematical realm (of structures, individual objects, or simply an independent reality). This comparison shows that each is as problematic as the other.

For example, in fictionalism, abstract objects – like sets and numbers – are exchanged for a primitive notion of logical possibility (and uncountably many points and regions; Shapiro, 1997, 223), but Shapiro notes that there are direct, trivial translations from the fictionalists' language into the realists' and vice versa. The translation from realism to fictionalism is simply a matter of inserting modal operators in appropriate places, and perhaps conjoining axioms. For example, the translation of a sentence (x) would be of the form ◊(alpha & (x)*) where alpha is a conjunction of axioms from the background mathematical theory, and (x)* is a variant of x. For the converse, simply replace possibility with satisfiability. A subformula of the form 'phi is possible' becomes 'phi is satisfiable'. Modal structuralism is similarly tackled on page 229, Charles Chihara's 'constructibility' quantifier is tackled on page 230, and George Boolos' plural quantification is tackled on page 233. Boolos proposes plural quantifiers as a way of interpreting second-order quantifiers – that is, instead of reading these as saying 'there is a class' or 'there is a property', to read them as saying something like 'there are objects' or 'there are people'. But 'here too there are straightforward translations between the standard metalanguage, with classes and no plurals, and the classless language with plural quantifiers' (p. 234).

In short, as mentioned earlier, typically an anti-realist tries to replace abstract entities (numbers, functions, set, etc.) in mathematics with talk of something apparently more acceptable. But so long as such attempts also try to account for all or most of mathematics as it is practised, they wind up having to quantify over all of whatever they talk about (or quantify over); for example, they may use a plural quantifier, a constructibility quantifier, or an operator for logical possibility, or they may quantify over proofs. However, all of these are also abstract objects, so we can ask all the same questions of these replacements as we did of the realist's numbers, structures, and so on. And this is just exactly why we wind up with inter-translatability.

Given this situation, Shapiro concludes that we might as well opt for the real realm. This is primarily because this option minimises the 'ideology' needed to explain the realm in question (given that a real realm takes mathematics at face value), is the 'most perspicuous and natural' interpretation and, not least, explains why mathematics is done as though it is about independently real objects.

This is an issue for these sorts of anti-revisionist programs, of course, because whatever type of 'mental constructions' you might want to propose *instead* of mathematical-type objects, you still wind up needing to talk of *these* directly and, in comparably problematic ways, to account for mathematics as we *know it*. Haim Gaifman (2012) makes the same point when he says that (potential anti-realist) mathematicians ask questions about 'the provability of a sentence' or 'the consistency of a theory' as though such questions have definite answers.

But then they're talking about proof in just the same way that mathematical realists talk about other abstract mathematical stuff like numbers or sets.

So if we try to reduce all number/set talk to talk of constructible or apparently more accessible things like proofs, we don't avoid talk of mathematical things; in a sense, we just rename them.

I accept the arguments here and take it as true that modal anti-realism (where 'modal anti-realism' encompasses all the programs Shapiro reviews) is, in an important sense, equivalent to ante rem structuralism.

But the arguments show more than this. Again, equivalence goes both ways. Unless something more than a reduced ideology sets it apart from its anti-realist counterparts, ante rem structuralism inherits the features of modal anti-realism, including, for example, the particular burden of explaining why mathematics can legitimately be taken at face value at all – how, for example, its objects are still as bona fide as any objects in the natural or physical world. And these sorts of questions mean that the realist camp now also has to take on board precisely what Shapiro hoped to avoid – an increased ideology.

So the proposed 'trade off . . . between a vast ontology and an increased ideology' (Shapiro 1997, 218) has more ramifications than are at first apparent. The trade-off does, as Shapiro claims, mean that any possible reality inherits all the problems of an actual reality, especially since a possible ontology is, in the end, no smaller or less problematic than its real counterpart.

But the trade-off also means that a real or actual ontology is, for all intents and purposes, equivalent to a possible ontology – unless something extra sets the former apart from the latter – and so is not (without further argument) real in quite the same way it intended, crucially including not as independent physical reality. That is, it is not easy to see how such an actual reality is in any way independent – either of the constructed or possible reality it is equivalent to or, by extension, of the quantifiers and concepts from which a possible ontology is derived.

3 Two Access Problems

In the introduction to his 1997 book, Shapiro outlines a number of desiderata for a philosophy of mathematics. These begin with the assertion that mathematical assertions ought to be taken literally, that is, 'at face value' (p. 3). This desideratum, for Shapiro, suggests realism in mathematics for two reasons. First, mathematical assertions tend to talk about mathematical objects as though they exist. The second is that scientific assertions tend to talk about scientific objects as though they exist, and scientific language is effectively inseparable from mathematical language.

Specifically, since model theoretic semantics applies to ordinary and scientific language, the argument is that this same model theoretic semantics ought to apply to mathematical language.

This means that the initial desideratum boils down to two separate desiderata. The first – realism in ontology – arises from the fact that model theoretic semantics has the singular terms of its language denoting objects and the variables ranging over a domain of discourse. Taking mathematics at face value, then, means taking mathematical objects to exist. The second – realism in truth-value – arises from the fact that the model theoretic framework attributes to each well-formed, meaningful sentence a determinate and non-vacuous truth-value, either true or false. Thus, the primary requirement that Shapiro places on his own account is 'to develop an epistemology for mathematics while maintaining the ontological and semantic commitments [above]' (p. 4).

Since then, Shapiro has reaffirmed this realism (2011, 130):

> According to my *ante rem* structuralism, the subject-matter of a branch of mathematics is a structure, or class of structures, that exist objectively, independently of the community of mathematicians and scientists, their minds, languages, forms of life, etc ... So *ante rem* metaphysics of mathematics.

Frege gave similar arguments for the existence of mathematical objects and for the truth of mathematical theorems (see Linnebo (2018), section 2.1). Frege's argument points out that mathematical language seems to refer to objects in much the same way that ordinary language does. This argument rests on the notion that certain subject-predicate phrases in ordinary, or *natural* language can only apparently be tested for truth or falsity – and perhaps can only be understood at all if any given hearer or translator supposes that there is such a subject to be found in the 'real' world, and that the properties of that subject can be discovered through scientific or empirical investigation. For example, the truth of a claim that 'Dogs are four-legged' can be investigated by referring to actual dogs and their supposed properties. Frege points out that given a great many mathematical claims have the same semantic structure (e.g. 'Natural numbers are divisible'), the burden should be on anyone insisting these two types of claims are to be treated differently, rather than on anyone who treats them similarly – that is, rather than on anyone who treats both as referring to objects in the world to be investigated as to their supposed properties.

Of course, the hitch here is that the idea that we can 'discover' mathematical truths in the same way that we can discover empirical or physical truths begs the question – or the problem – of access.