Algebraic Geometry

A Celebration of Emma Previato's 65th Birthday

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1 A Word from the Editors

These two volumes celebrate and honor Emma Previato, on the occasion of her 65th birthday. The present volume consists of 16 articles in algebraic geometry and its surrounding fields, emphasizing the connections to integrable systems which are so central to Emma's work. The companion volume focuses on Emma's interests within integrable systems. The articles were contributed by Emma's coauthors, colleagues, students, and other researchers who have been influenced by Emma's work over the years. They present a very attractive mix of expository articles, historical surveys and cutting edge research.

Emma Previato is a mathematical pioneer, working in her two chosen areas, algebraic geometry and integrable systems. She has been among the first women to do research, in both areas. And her work in both areas has been deep and influential. Emma received a Bachelor's degree from the University of Padua in Italy, and a PhD from Harvard University under the direction of David Mumford in 1983. Her thesis was on hyperelliptic curves and solitons. The work on hyperelliptic curves has evolved and expanded into Emma's life-long interest in algebraic geometry, which is the subject of the present volume. The work on solitons has led to her ongoing research on integrable systems, which is the subject of the first volume.

Emma Previato has been a faculty member at the Department of Mathematics at Boston University since 1983. She has published nearly a hundred research articles, edited six books, and directed seven PhD dissertations. Her broader impact extends through her renowned teaching and her extensive mentoring activities. She runs AFRAMATH, an annual outreach symposium, and works tirelessly on several on- and off-campus mentoring programs. She has also founded and has been leading the activities of the Boston University chapters of MAA and AWM. She serves on numerous advisory boards.

Algebraic geometry is an ancient subject that has been at the center of mathematical research for at least the past two centuries, attracting many of the greatest mathematical minds. Previato's work has made important contributions in many directions within algebraic geometry, and especially so in areas that connect with the theory of integrable systems. We review some of her accomplishments in section 2 below.

We tried to collect in this volume a broad range of articles covering most of these areas. We summarize these contributions in section 3.

We would like to quote David Mumford, Emma's PhD advisor, who wrote in the context of this volume:

"One of the greatest pleasures in a Professor's life is when one of your PhD students goes on to become a world leader in the area of their thesis. In fact, all my seven female PhD students have done extremely well. Emma was the last in algebraic geometry and it makes me sad that when my own research shifted, I have not followed hers. But reading the list of her papers and their topics completely blows me away. The scope of her work is astonishing: it is the whole area where algebraic geometry meets differential equations and the special functions and spectral curves (and surfaces) that arise. Congratulations Emma on your 65th birthday, you have come such a long way since we shared gelato in Harvard Square and I am deeply happy to see what you have done."

The editors and many of the authors have enjoyed years of fruitful interactions with Emma Previato. We all join in wishing her many more years of health, productivity, and great mathematics.

2 Emma Previato's Contributions

Emma Previato works in different areas, using methods from algebra, algebraic geometry, mechanics, differential geometry, analysis, and differential equations. The bulk of her research belongs to integrable equations. She is noted for often finding unexpected connections between integrability and many other areas, often including various branches of algebraic geometry.

Early Activity

As an undergraduate at the University of Padua, Italy, Emma wrote a dissertation on group lattices, followed by six journal publications [1, 2, 3, 4, 5, 6]. With methods from algebra, initiated by Dedekind in the 19th century, this area's goal is to relate the group structure to the lattice of subgroups, and provide classifications for certain properties: an excellent overview is the article by Freese [7], a review of the definitive treatise by R. Schmidt, where results from all of Emma's papers are used to give one example, a lattice criterion for a finitely generated group to be solvable.

PhD Thesis and Main Area

Emma's thesis [8], submitted at Harvard in 1983 under the supervision of D.B. Mumford [9], is still her most cited paper. Her thesis advisor was among the pioneers of this beautiful area, integrable equations, which grew and unified disparate parts of mathematics over the next twenty years, and is still very active. Emma's original tool for producing exact solutions to large classes of nonlinear PDEs, the Riemann theta function, remained one of her main interests.

Theta Functions

She later pursued more theoretical aspects of special functions, such as Prym theta functions [10, 11, 12, 13, 14] also surprisingly related to numerical results in conformal field theory, the Schottky problem [15], and Thetanulls [16].

Algebraically Completely Integrable Systems

The area of integrable PDEs is surprisingly related to algebraically completely integrable Hamiltonian systems, or ACIS, in the sense that algebro-geometric (aka finite-gap) solutions of integrable hierarchies linearize on Abelian varieties, which can be organized into angle variables for an ACIS over a suitable base, typically a subset of the moduli space of curves whose Jacobian is the fiber [12, 17]. Thanks to this discovery, the area integrates with classical geometric invariant theory, surface theory, and other traditional studies of algebraic geometry. With the appearance of the moduli spaces of vector bundles and Higgs bundles over a curve, at the hands of N. Hitchin in the 1980s, large families of ACIS were added to the examples, as well as theoretical algebro-geometric techniques. In [13, 18, 19, 20], Emma took up the challenge of generalizing the connection between ACIS and integrable hierarchies to curves beyond hyperelliptic. In [21], the families of curves are organized as divisors in surfaces.

Higher Rank and Higher-Dimensional Spectra

On the PDE side, the challenges were of two types. When the ring of functions on the (affine) spectral curve can be interpreted as differential operators with a higher-dimensional space of common eigenfunctions, the fiber of the integrable system is no longer a Jacobian: it degenerates to a moduli space of higher-rank vector bundles, possibly with some auxiliary structures [22]. Neither the PDEs nor the integrable systems have been made explicit in higher rank in general. Some cases, however, are worked out in [23, 24, 25, 26, 27].

The other challenge is to increase the dimension of the spectral variety, for example from curve to surface. Despite much work, this problem too has arguably no explicit solution in general. An attempt to set up a general theory over a multi-dimensional version of the formal Universal Grassmann Manifold of Sato which hosts all linear flows of solutions of integrable hierarchies, is given in [28], and more concrete special settings are mentioned below, under the heading of "Differential Algebra".

Special Solutions: Coverings of Curves

An important aspect of theta functions is their reducibility, a property whose investigation goes back to Weierstrass and his student S. Kowalevski. Given their special role in integrability, reducible theta functions are invaluable for applied mathematicians to approximate solutions, or even derive exact expressions and periods in terms of elliptic functions. To the algebro-geometric theory of Elliptic Solitons, initiated by I.M. Krichever and developed by A. Treibich and his thesis supervisor J.-L. Verdier, Emma contributed [29, 30, 31, 32, 33, 34, 11, 35], while [36, 37] generalize the reduction to hyperelliptic curves or Abelian subvarieties. More general aspects of elliptic (sub)covers are taken up in [38].

Another type of special solution is the one obtained by self-similarity [39]; the challenge here is to find an explicit relationship between the PDE flows and the deformation in moduli that obeys Painlevé-type equations: this is one reason why Emma's work has turned to a special function which is associated to Riemann's theta function but only exists on Jacobians: the sigma function (cf. the eponymous section below).

Poncelet and Billiards: Generalizing ACIS

Classical theorems of projective geometry can be generalized to ACIS [40, 41], while the challenge of matching them with integrable hierarchies is still ongoing [42].

Generalizing ACIS: Hitchin Systems

Explicit Hamiltonians for the Hitchin system are only available in theory: they are given explicit algebraic expression in [43] (cf. also [44], which led to work on the geometry of the moduli space of bundles [45]). An explicit integration in terms of special functions leads to the problem of non-commutative theta functions [46].

Differential Algebra

Differential Algebra is younger than Algebraic Geometry, but it has many features in common. Mumford gives credit to J.L. Burchnall and T.W. Chaundy for the first spectral curve, the Spectrum of a commutative ring of differential operators [47]. This is arguably the reason behind algebro-geometric solutions to integrable hierarchies. On the differential-algebra setting, Emma published [48, 49], connecting geometric properties of the curve with differential resultants, a major topic of elimination theory which is currently being worked out [50, 51] and naturally leads to the higher-rank solutions: their Grassmannian aspects are taken up in [52, 53, 54, 55, 56] the higher-dimensional spectral varieties are addressed in [57]. Other aspects of differential algebra are connected to integrability in [58] (the action of an Abelian vector field on the meromorphic functions of an Abelian variety) and [59] (a p-adic analog); in [60], the deformations act on modular forms.

The Sigma Function

Klein extended the definition of the (genus-one) Weierstrass sigma function to hyperelliptic curves and curves of genus three. H.F. Baker developed an in-depth theory of PDEs satisfied by the hyperelliptic sigma function, which plays a key role in recent work on integrable hierarchies (KdV-type, e.g.). Beginning in the 1990s, this theory of Kleinian sigma functions was revisited, originally by V.M. Buchstaber, V.Z. Enolskii, and D.V. Leykin, much extended in scope, eventually to be developed for "telescopic" curves (a condition on the Weierstrass semigroup at a point). Previato goes beyond the telescopic case in [61, 62], while she investigates the higher-genus analog of classical theorems in [63, 64, 65, 66, 67, 68, 69, 70] and their connections with integrability in [71] and [72], which gives the first algebro-geometric solutions to a dispersionless integrable hierarchy. It is not a coincidence that its integrable flow on the Universal Grassmann Manifold 'cut across' the Jacobian flows of traditional hierarchies, and this is where the two variables of the sigma function (the Jacobian, and the modular ones) should unite to explain the mystery of the Painlevé equations.

Algebraic Coding Theory

Emma's primary contribution to this area is through mentoring undergraduate and graduate thesis or funded-research projects. In fact, this research strand began at the prompting of students in computer science who asked her to give a course on curves over fields of prime characteristic, which she ran for years as a vertically-integrated seminar. Together with her PhD student Drue Coles, she published research papers pursuing Trygve Johnsen's innovative idea of error-correction for Goppa codes implemented via vector bundles [73, 74, 75], then she pursued overviews and extensions of Goppa codes to surfaces [76].

Other

Emma edited or co-edited four books [77, 78, 79, 80]. In addition to book and journal publication, Emma published reviews (BAMS, SIAM), entries in mathematical dictionaries or encyclopaedias, teaching manuals and online research or teaching materials; she also published on the topic of mentoring in the STEAM disciplines.

3 Articles in this Volume

Several of the articles in this volume explore arithmetic aspects of integrable systems:

Buium's article is an exposition of a massive body of work establishing arithmetic analogues of ODEs that preserve the underlying geometry, and exploring arithmetic versions of integrable systems.

The article by Dubrovin explores **Q**-structures on spectral curves and their Jacobians. Under some natural assumptions he proves that certain combinations of logarithmic derivatives of the Riemann theta-function of an arbitrary order ≥ 3 all take rational values at the point of the Jacobi variety specified by the line bundle of common eigenvectors.

Kovacs extends to arbitrary characteristic two theorems of Sommese regarding Abelian varieties: that an Abelian variety cannot be an ample divisor in a smooth projective variety, and that a cone over an Abelian variety of dimension at least two is not smoothable.

Horozov proves that every multiple Dedekind zeta value over any number field K is a period of a mixed Tate motive. Moreover, if K is a totally real number field, then we can choose a cone C so that every multiple Dedekind zeta associated to the pair (K;C) is unramified over the ring of algebraic integers in K. (Multiple Dedekind zeta values associated to a number field K and a cone C were defined previously by the author as number theoretic analogues of multiple zeta values.)

A second group of articles explores various aspects of curves or Riemann surfaces: The points of order 2 in the Jacobian of a curve form a group, and the halves of any divisor of degree 0, e.g. the difference $P - Q$ of two points, form a torsor under this group. Zarhin's work describes this torsor explicitly when the curve is hyperelliptic and the point P is a Weierstrass point.

Komeda and Matsutani investigate curves given by a natural generalization of Weierstrass' equation and show the Jacobi inversion formulae of the strata of its Jacobian.

Spera presents a survey of some recent topological applications of Riemann surface theory, and especially of the associated theta functions, to geometric quantization in mathematical physics. These include classical and quantum monodromy of 2d-integrable systems, the construction of unitary Riemann surface braid group representations, and the noncommutative version of theta functions due to A. Schwarz.

An emphasis on special functions attached to a Riemann surface is the common theme of the next group:

Buchstaber, Enolski, and Leykin consider the dependence of multi-variable sigma function of genus g hyperelliptic curves on both Jacobian variables and parameters of the curve. They develop new representations of periods in terms of theta-constants.

Korotkin contributed a review of the role played by Bergman tau-functions in various areas: theory of isomonodromic deformations, solutions of Einstein's equations, theory of Dubrovin-Frobenius manifolds, geometry of moduli spaces, and spectral theory of Riemann surfaces. These tau-functions are natural generalizations of Dedekinds eta-function to higher genus.

Cogdell, Jorgenson, and Smajlov construct a generalization of the Eisenstein elliptic series attached to finite volume Riemann surfaces. Their input is a pair (X, D), where X is a smooth, compact, projective Kahler variety and D a divisor of a holomorphic form F, assumed smooth up to codimension two. The original case is recovered when X is the quotient of a symmetric space.

Other algebraic aspects of integrable systems appear in the next four papers. In two of them the emphasis is on the Kummer surfaces that appear in genus-2 systems.

Bates and Churchill focus on Lax pairs. These are useful, but often hard to find. The authors describe an elementary systematic construction which has proven successful in several contexts of practical interest. Specific examples are presented.

Retakh, Rubtsov, and Sharygin present their theory of noncommutative cross-ratios, Schwarz derivatives and their connections and relations to the operator cross-ratio. They apply the theory to noncommutative elementary geometry and relate it to noncommutative integrable systems.

The work of Clingher and Malmendier determines normal forms for the Kummer surfaces associated with Abelian surfaces of several polarization of types. It also produces explicit formulas for coordinates and moduli parameters in terms of Theta functions of genus two.

Francoise and Tarama wrote an expository article about the completely integrable Hamiltonian system of the Clebsch top under a special condition introduced by Weber. This led them to Kummer surfaces and to linearization on genus 2 Jacobians.

Finally, two works connect with other aspects of Emma's research:

Carvalho and Neumann present both a survey and some new results in coding theory, extending the well-known work of Delsarte, Goethals, and Mac Williams to a class of Reed-Muller type codes defined on a product of (possibly distinct) finite field the same characteristic.

Schmidt's work aims at a lattice-theoretic characterization of various groups. First he does this for some classes of infinite soluble groups. Then for a finite group G he tries to determine in its subgroup lattice L(G), the Fitting length of G, and properties defined by arithmetical conditions.

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