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Infinite Planar Graphs with Non-negative Combinatorial Curvature

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Abstract

In this chapter, we survey some results on infinite planar graphs with non-negative combinatorial curvature, related to the total curvature, the number of vertices with positive curvature and the automorphism group.

1.1 Introduction

The combinatorial curvature for planar graphs was introduced by Nevanlinna, Stone, Gromov, and Ishida [Nev70, Sto76, Gro87, Ish90] respectively, which resembles the Gaussian curvature for smooth surfaces. Many interesting geometric and combinatorial results have been obtained under such curvature conditions since then (see, e.g., [Ż97, Woe98, Hig01, BP01, HJL02, LPZ02, HS03, SY04, RBK05, BP06, DM07, CC08, Zha08, Che09, Kel10, KP11, Kel11, Oh17, Ghi17]).

Let (V, E) be a (possibly infinite) locally finite, undirected simple graph with the set of vertices V and the set of edges E. It is called *planar* if it can be topologically embedded into the sphere $\mathbb{S}^2$ or the plane $\mathbb{R}^2$, where we distinguish $\mathbb{S}^2$ with $\mathbb{R}^2$ while they are identified in the theory of finite planar graphs. We write $G = (V, E, F)$ for the cellular complex structure of a planar graph induced by the embedding where F is the set of faces, i.e., connected components of the complement of the embedding image of the graph (V, E) in $\mathbb{S}^2$ or $\mathbb{R}^2$. We say that a planar graph G is a *planar tessellation* if the following hold (see, e.g., [Kel11]):

(i) Every face is homeomorphic to a disc whose boundary consists of finitely many edges of the graph.

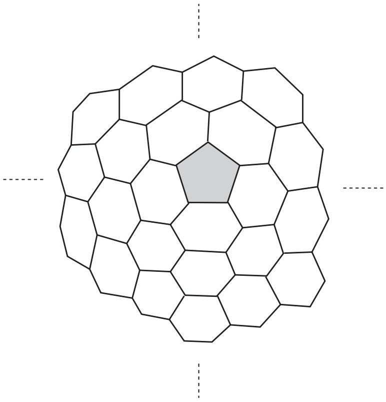

Figure 1.1 A planar graph G consists of a pentagon and infinitely many hexagons

(ii) Every edge is contained in exactly two different faces.
(iii) For any two faces whose closures have non-empty intersection, the intersection is either a vertex or an edge.

In this chapter, we only consider planar tessellations (see Figure 1.1 for an example) and call them *planar graphs* for the sake of simplicity. For a planar tessellation, it is finite (infinite resp.) if and only if it embeds into $\mathbb{S}^2$ ($\mathbb{R}^2$ resp.).

We say that a vertex x is incident to an edge e, denoted by $x \prec e$, (similarly, an edge e is incident to a face σ, denoted by $e \prec \sigma$; or a vertex x is incident to a face σ, denoted by $x \prec \sigma$) if the former is a subset of the closure of the latter. Two vertices x and y are called 'neighbours' if there is an edge e such that $x \prec e$ and $y \prec e$, in this case denoted by $x \sim y$. We denote by $\deg(x)$ the degree of a vertex x, i.e., the number of neighbours of a vertex x, and by $\deg(\sigma)$ the degree of a face σ, i.e., the number of edges incident to a face σ (equivalently, the number of vertices incident to σ). We always assume that for any vertex x and face σ,

$$\deg(x) \geq 3, \ \deg(\sigma) \geq 3.$$

We denote by

$$(\deg(\sigma_1), \deg(\sigma_2), \cdots, \deg(\sigma_N))$$

the pattern of a vertex x where $N = \deg(x)$, $\{\sigma_i\}_{i=1}^N$ are the faces which x is incident to, and $\deg(\sigma_1) \leq \deg(\sigma_2) \leq \cdots \leq \deg(\sigma_N)$.

Given a planar graph $G = (V, E, F)$, one may canonically endow its ambient space $\mathbb{S}^2$ or $\mathbb{R}^2$ with a piecewise flat metric as follows: assign each edge length one, replace each face by a regular Euclidean polygon of side length one with same facial degree, and glue these polygons along the common edges. The ambient space equipped with the induced metric constructed above is called the *regular polyhedral surface* of G, denoted by $S(G)$. In the following, we always call it the *polyhedral surface* for the sake of brevity. For a planar graph G, the *combinatorial curvature* at the vertex is defined as

$$\Phi(x) = 1 - \frac{\deg(x)}{2} + \sum_{\sigma \in F : x \prec \sigma} \frac{1}{\deg(\sigma)}, \quad x \in V. \tag{1.1.1}$$

In this chapter, we mean by the curvature of a planar graph the combinatorial curvature of it for simplicity. It turns out that the curvature of a planar graph is given by the generalized Gaussian curvature of the polyhedral surface $S(G)$ up to some normalization. Note that for the polyhedral surface $S(G)$ it is locally isometric to a flat domain in $\mathbb{R}^2$ near any interior point of an edge or a face, while it might be non-smooth near the vertices. As a metric surface, the generalized Gaussian curvature K of $S(G)$ vanishes at smooth points and can be regarded as a measure concentrated on the isolated singularities, i.e., on vertices. One can show that the mass of the generalized Gaussian curvature at each vertex x is given by $K(x) = 2\pi - \Sigma_x$, where Σ_x denotes the total angle at x in the metric space $S(G)$ (see [Ale05]). Moreover, by direct computation one has $K(x) = 2\pi\Phi(x)$, where the curvature $\Phi(x)$ is defined in (1.1.1). Hence, one can show that a planar graph G has non-negative curvature if and only if the polyhedral surface $S(G)$ is a generalized convex surface in the sense of Alexandrov (see [BGP92, BBI01, HJL15]). Furthermore, the polyhedral surface $S(G)$ can be isometrically embedded into $\mathbb{R}^3$ as a boundary of a compact or non-compact convex polyhedron by Alexandrov's embedding theorem ([Ale05]); see Figure 1.2 for an embedded image of $S(G)$ of the planar graph G in Figure 1.1.

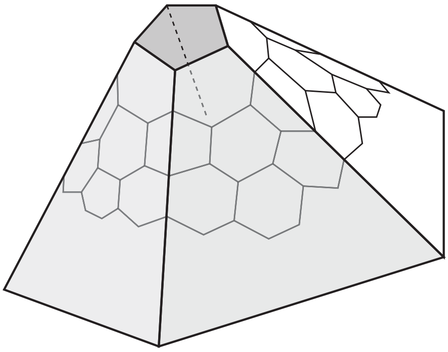

Figure 1.2 The isometric embedding of $S(G)$ of the planar graph G in Figure 1.1

In this chapter, we study planar graphs with non-negative curvature. We introduce two classes of planar graphs with positive or non-negative curvature as follows:

- $\mathcal{PC}_{>0} := \{G : \Phi(x) > 0, \forall x \in V\}$ is the class of planar graphs with positive curvature everywhere.
- $\mathcal{PC}_{\geq 0} := \{G : \Phi(x) \geq 0, \forall x \in V\}$ is the class of planar graphs with non-negative curvature everywhere.

We review some known results on the class $\mathcal{PC}_{>0}$. Stone [Sto76] first proved a Myers-type theorem: a planar graph with the curvature bounded below uniformly by a positive constant is a finite graph. Higuchi proposed a stronger conjecture that any $G \in \mathcal{PC}_{>0}$ is a finite graph (see [Hig01, Conjecture 3.2]). This is certainly wrong for smooth surfaces since there are many non-compact convex surfaces in $\mathbb{R}^3$, which have positive curvature everywhere. However, for a planar graph it is hopefully true by the combinatorial restriction of regular polygons as its faces. DeVos and Mohar [DM07] proved the conjecture by showing a generalized Gauss–Bonnet formula (see [SY04] for the case of cubic graphs).

For any finite planar graph $G \in \mathcal{PC}_{\geq 0}$, in particular any $G \in \mathcal{PC}_{>0}$, by Alexandrov's embedding theorem its polyhedral surface $S(G)$ can be isometrically embedded into $\mathbb{R}^3$ as a boundary of a convex polyhedron (see, e.g., [Ale05]). From this point of view, we obtain many examples for the class $\mathcal{PC}_{>0}$, e.g., the 1-skeletons of 5 Planotic solids, 13 Archimedean solids, and 92 Johnson solids. Any of them has regular Euclidean polygons as its faces in its embedded image in $\mathbb{R}^3$. Note that these are all examples of planar graphs in $\mathcal{PC}_{>0}$ whose faces of the embedded image in $\mathbb{R}^3$ are regular polygons (see [Joh66, Zal67]). Besides these, the class $\mathcal{PC}_{>0}$ contains many other examples, such as an example of 138 vertices constructed by Réti, Bitay, and Kosztolányi [RBK05], examples of 208 vertices by Nicholson and Sneddon [NS11], Ghidelli [Ghi17], and Oldridge [Old17], which cannot be realized as the boundary of a convex polyhedron whose faces are regular polygons. In fact, although any face of $G \in \mathcal{PC}_{>0}$ is isometric to a regular polygon in $S(G)$, it may split into several pieces of non-coplanar faces in the embedded image of $S(G)$ as the boundary of a convex polyhedron in $\mathbb{R}^3$.

There are two special families of graphs in $\mathcal{PC}_{>0}$ called *prisms* and *antiprisms*, both consisting of infinite many examples (see, e.g., [DM07]). Besides them, DeVos and Mohar [DM07] proved that there are only finitely many graphs in $\mathcal{PC}_{>0}$ and proposed the following problem to find out the largest graph among them.

Problem 1.1.1 ([DM07]) *What is the number*

$$C_{\mathbb{S}^2} := \max_{G=(V,E,F)} \sharp V,$$

where the maximum is taken over graphs in $\mathcal{PC}_{>0}$, which are not prisms or antiprisms, and $\sharp V$ denotes the cardinality of V?

On the one hand, as some examples of 208 vertices in $\mathcal{PC}_{>0}$ have been constructed in [NS11, Ghi17, Old17], we have the lower bound estimate that $C_{\mathbb{S}^2} \geq 208$. On the other hand, DeVos and Mohar [DM07] initiated to use the discharging methods to obtain the upper bound estimate $C_{\mathbb{S}^2} \leq 3444$. The discharging methods were adopted in the proof of the four-colour theorem in the literature (see [AH77, RSST97]). The upper bound was later improved to $C_{\mathbb{S}^2} \leq 380$ by Oh [Oh17]. By a delicate argument, Ghidelli [Ghi17] showed that $C_{\mathbb{S}^2} \leq 208$, which completely solves DeVos and Mohar's problem that $C_{\mathbb{S}^2} = 208$.

Next, we consider the class of planar graphs with non-negative curvature, i.e., $\mathcal{PC}_{\geq 0}$, which turns out to be much larger than $\mathcal{PC}_{>0}$ and contains many interesting examples. The class of $\mathcal{PC}_{>0}$ consists of essentially finite many examples, while the class $\mathcal{PC}_{\geq 0}$ contains infinitely many examples of different combinatorial types. A fullerene is a finite cubic planar graph whose faces are either pentagon or hexagon. There are plenty of examples of fullerenes which are important in the real-world applications, to cite a few examples [KHO+85, Thu98, BD97, BGM12, BE17a, BE17b]. Note that any fullerene is a planar graph with non-negative curvature. As shown by Thurston [Thu98], the number of combinatorial types of fullerenes with N hexagons grows as N^9 as $N \to \infty$. Besides these examples of finite graphs, there are plenty of examples of infinite graphs. Any planar tiling with regular polygons as tiles (see, e.g., [GS89, Gal09]) is in the class $\mathcal{PC}_{\geq 0}$. Note that there are infinitely many such planar tilings, for which only a few examples with symmetry can be classified. These motivate us to investigate the general structure of planar graphs in the class $\mathcal{PC}_{\geq 0}$.

1.2 Total Curvature of Planar Graphs with Non-negative Curvature

For a smooth non-compact surface with absolutely integrable Gaussian curvature, its total curvature encodes the global geometric information of the space, e.g., the boundary at infinity (see [SST03]). For example, the total curvature of a convex surface in $\mathbb{R}^3$ describes the apex angle of the cone at infinity of the surface, which is useful to study global geometric and analytic properties of the surface, such as harmonic functions and heat kernels, following [CM97b, Xu14]. For planar graphs with non-negative curvature G, we denote by

$$\Phi(G) := \sum_{x\in V} \Phi(x)$$

the total curvature of G whenever the summation converges absolutely. In case of finite graphs, the Gauss–Bonnet theorem reads as (see, e.g., [DM07])

$$\Phi(G) = 2. \tag{1.2.1}$$

For an infinite planar graph $G \in \mathcal{PC}_{\geq 0}$, the Cohn-Vossen type theorem, proven by [DM07, Theorem 1.3] or [Che09, Theorem 1.6], yields that

$$\Phi(G) \leq 1. \tag{1.2.2}$$

This means that for any infinite $G \in \mathcal{PC}_{\geq 0}$, the total curvature of G satisfies

$$0 \leq \sum_{x\in V} \Phi(x) \leq 1.$$

In this section, we study all possible values of total curvature of infinite planar graphs with non-negative curvature, i.e., the following set

$$\{\Phi(G) : \text{G infinite}, G \in \mathcal{PC}_{\geq 0}\}. \tag{1.2.3}$$

As is well known in Riemannian geometry that for any real number $0 \leq a \leq 2\pi$, there is a convex surface whose total curvature is given by a. Hence, the above set for non-compact convex surfaces turns out to be an interval in the continuous setting. However, combinatorial structure of planar graphs with non-negative curvature gives us more information and restrictions for the set (1.2.3).

For any $G = (V, E, F) \in \mathcal{PC}_{\geq 0}$, we denote by

$$T_G := \{v \in V : \Phi(x) > 0\} \tag{1.2.4}$$

the set of vertices with positive curvature, and by

$$D_G := \sup_{\sigma\in F} \deg(\sigma) \tag{1.2.5}$$

the maximal facial degree of G. Chen and Chen [CC08, Che09] proved an interesting result that the set of vertices with positive curvature in a planar graph with non-negative curvature is a finite set. Hence, the supremum in (1.2.5) is in fact the maximum.

Theorem 1.2.1 (Chen and Chen) *For any* $G \in \mathcal{PC}_{\geq 0}$, T_G *is a finite set.*

This result makes our combinatorial setting distinguished from the Riemannian setting. Note that there are many non-compact convex surfaces with positive curvature everywhere, e.g., the elliptic paraboloid, i.e., the revolution surface of the graph $y = x^2$ with respect to the z axis in $\mathbb{R}^3$.

Moreover, if the maximal facial degree D_G of $G \in \mathcal{PC}_{\geq 0}$ is at least 43, then G has rather special structure, analogous to the prisms or antiprisms in the finite case (see [HJL15] or Theorem 1.3.2 in this chapter). In that case, one gets $\Phi(G) = 1$. Hence, for our purposes to understand the set (1.2.3), it suffices to consider planar graphs G with $D_G \leq 42$. Note that there are finitely many vertex patterns, consisting of faces of degree at most 42, with positive curvature (see Table 1.1 in the Appendix). Then one is ready to see that the set (1.2.3) is a discrete subset in [0, 1] (see, e.g., [HS17b, Proposition 2.3]).

T. Réti [HL16, Conjecture 2.1] was motivated to determine the following value

$$\tau_1 := \inf\left\{\Phi(G) : G \in \mathcal{PC}_{\geq 0}, \Phi(G) > 0\right\},$$

which is called the *first gap of total curvature* for infinite planar graphs in the class of $\mathcal{PC}_{\geq 0}$. He suggested that $\tau_1 = \frac{1}{6}$ and the minimum is attained by the graph consisting of a pentagon and infinitely many hexagons, which is a kind of infinite fullerene (see Figure 1.1). In [HS17a], we give an answer to Réti's problem.

Theorem 1.2.2 (Theorem 1.3 in [HS17a])

$$\tau_1 = \frac{1}{12}.$$

A planar graph $G \in \mathcal{PC}_{\geq 0}$ *satisfies* $\Phi(G) = \frac{1}{12}$ *if and only if the polyhedral surface* $S(G)$ *is isometric to either*

(a) a cone with the apex angle $\theta = 2 \arcsin \frac{11}{12}$, *or*
(b) a 'frustum' with a hendecagon base (see Figure 1.3).

The proof strategy is straightforward and involves tedious case studies. For a vertex with positive curvature, if the curvature of the vertex is less than $\frac{1}{12}$, then we try to find some nearby vertices with positive curvature such that the sum of these curvatures is at least $\frac{1}{12}$ and prove the results case by case. Note that there are examples of graphs in $\mathcal{PC}_{\geq 0}$ whose total curvature attains the first gap $\frac{1}{12}$ (see Figure 1.4 and [HS17a] for more examples). Although graph structures of infinite planar graphs attaining the first gap of total curvature could be as complicated as planar tilings (see [HS17a]) we are able to classify

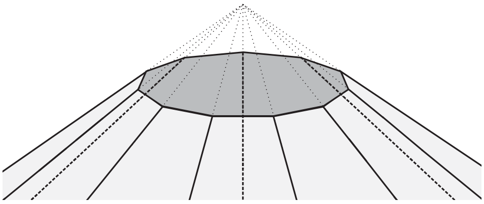

Figure 1.3 A ‘frustum’ with a hendecagon base

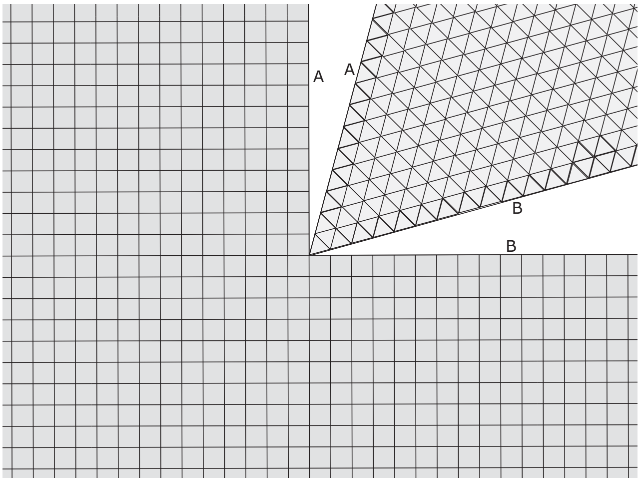

Figure 1.4 This is an example of total curvature $\frac{1}{12}$, where the half lines with same labels, A or B, are identified

metric structures of polyhedral surfaces for such planar graphs in the above theorem.

Inspired by Réti’s question, it will be interesting to know other values in the set (1.2.3). Using Chen and Chen’s result, Theorem 1.2.1, and the Gauss–Bonnet theorem for compact subsets with boundary, we are able to determine all possible total curvatures in the class $\mathcal{PC}_{\geq 0}$.

Theorem 1.2.3 (Theorem 1.1 in [HS17b]) *The set of all values of total curvature of infinite planar graphs with non-negative curvature* (1.2.3) *is given by*

$$\left\{\frac{i}{12} : 0 \leq i \leq 12, i \in \mathbb{Z}\right\}.$$

As a corollary, we also obtain that $\tau_1 = \frac{1}{12}$, which provides an alternative proof to Réti’s problem. Moreover, as the part of the theorem, one may construct planar graphs with non-negative curvature whose total curvatures attain

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all values listed above (see [HS17b]). We sketch the proof of the theorem as follows: by Theorem 1.2.1, we know that T_G is a finite set. We choose a sufficiently large compact subset $K \subset S(G)$, homeomorphic to a closed disc, such that it contains T_G and consists of faces in F. Note that the vertices on the boundary of K have vanishing curvature, so that their patterns appear in the list of 17 possible patterns in Table 1.2 in the Appendix. By some combinatorial restrictions, one can further exclude several patterns from the list and conclude that any vertex on the boundary is incident to a triangle, a square, a hexagon, an octagon, or a dodecagon. Then using the Gauss–Bonnet formula on K, we may prove the theorem. Similar proof strategies apply to the problems on the total curvature of a planar graph with boundary, i.e., a graph embedded into the disc or a half plane (see [HS17b]).

Although we crucially use the finiteness structure of T_G in the proof of Theorem 1.2.3, we don't know much about the structure of the subset T_G which still lies in a black box. By a byproduct of the proof of Theorem 1.2.2, we can show that for $G \in \mathcal{PC}_{\geq 0}$, the induced subgraph on T_G has at most 14 connected components. It was conjecturally at most 12 (see [HS17a, Conjecture 5.2]).

1.3 The Vertices of Positive Curvature in Planar Graphs with Non-negative Curvature

In this section, we survey some results on the set of vertices with positive curvature in planar graphs with non-negative curvature. For any finite (infinite resp.) $G \in \mathcal{PC}_{\geq 0}$, Alexandrov's embedding theorem [Ale05] yields that an isometric embedding of the polyhedral surface $S(G)$ into $\mathbb{R}^3$ as a boundary of a compact (non-compact resp.) convex polyhedron. The set T_G serves as the set of the vertices/corners of the convex polyhedron, so that much geometric information of the polyhedron is contained in T_G. We are interested in the structure of the set T_G.

By the solution to DeVos and Mohar's problem [Ghi17], besides the prisms and antiprisms the largest number of vertices in a finite graph in $\mathcal{PC}_{>0}$ is 208. We would like to study analogous problems for planar graphs in $\mathcal{PC}_{\geq 0}$. We define some analogues to prisms and antiprisms in the class $\mathcal{PC}_{\geq 0}$.

Definition 1.3.1 *We call a planar graph $G = (V, E, F) \in \mathcal{PC}_{\geq 0}$ a prism-like graph if either*

(1) G is an infinite graph and $D_G \geq 43$, where D_G is defined in (1.2.5)*, or*
(2) G is a finite graph and there are at least two faces with facial degree at least 43.

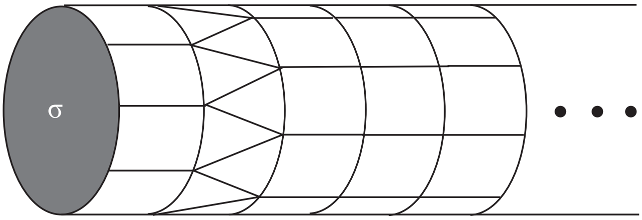

Figure 1.5 A half flat-cylinder in $\mathbb{R}^3$

By dividing hexagons into triangles, one may assume that there is no hexagon in G. Note that 'prism-like' graphs have rather special structures which can be completely determined by the following theorems. For any face σ, we denote by

$$\partial\sigma := \{x \in V : x \prec \sigma\}$$

the vertex boundary of σ.

Theorem 1.3.2 ([HJL15]) *Let $G = (V, E, F)$ be an infinite planar graph with non-negative curvature and $D_G \geq 43$. Then there is only one face σ of degree at least 43. Suppose that there is no hexagonal face. Then the set of faces F consists of σ, triangles or squares. Moreover,*

$$F = \sigma \cup (\cup_{i=1}^{\infty} L_i),$$

where L_i, $i \geq 1$, are sets of faces of the same type (triangle or square) which composite a band, i.e., an annulus, and is defined inductively: L_i is the next layer attaching to the previous layer L_{i-1} with $L_0 = \{\sigma\}$. $S(G)$ is isometric to the boundary of a half flat-cylinder in $\mathbb{R}^3$ (see Figure 1.5). Moreover, $\Phi(G) = 1$.

Theorem 1.3.3 ([HS18]) *Let $G = (V, E, F)$ be a finite prism-like graph. Then there are exactly two disjoint faces σ_1 and σ_2 of same facial degree at least 43. Suppose that there is no hexagonal face. Then the set of faces F consists of σ_1 and σ_2, triangles, or squares. Moreover,*

$$F = \sigma_1 \cup (\cup_{i=1}^{M} L_i) \cup \sigma_2,$$

where $M \geq 1$, and L_i, $1 \leq i \leq M$, are defined similarly as in Theorem 1.3.2. $S(G)$ is isometric to the boundary of a cylinder barrel in $\mathbb{R}^3$ (see Figure 1.6).

The following problem was proposed in [HL16] as an analogue to DeVos and Mohar's problem.