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Power Series in Fifteenth-Century Kerala

1.1 Preliminary Remarks

More than two and a half centuries before Isaac Newton discovered the sine and cosine series and James Gregory the arctan series, the Indian astronomer and mathematician Madhava (c. 1340–c. 1425) gave expressions for $\sin x$, $\cos x$, and $\arctan x$ as infinite power series.[1] Madhava's work may have been motivated by his studies in astronomy, since he concentrated mainly on the trigonometric functions. There appears to be no connection between the work of Madhava's school and that of Newton and other European mathematicians. In spite of this, the Keralese and European mathematicians shared some similar methods and results. Both were fascinated with transformation of series, though they used very different methods.

The mathematician-astronomers of medieval Kerala lived, worked, and taught in large family compounds called illams. Madhava, believed to have been the founder of the school, worked in the Bakulavihara illam in the town of Sangamagrama, a few miles north of Cochin. He was an Emprantiri Brahmin, then considered socially inferior to the dominant Namputiri (or Nambudri) Brahmin. This position does not appear to have curtailed his teaching activities; his most distinguished pupil was Paramesvara, a Namputiri Brahmin. No mathematical works of Madhava have been found, though three of his short treatises on astronomy are extant. The most important of these describes how to accurately determine the position of the moon at any time of the day. Other surviving mathematical works of the Kerala school attribute many very significant results to Madhava. Although his algebraic notation was almost primitive, Madhava's mathematical skill allowed him to carry out highly original and difficult research.

Paramesvara (c.1380–c.1460), Madhava's pupil, was from Asvattagram, about thirty-five miles northeast of Madhava's home town. He belonged to the Vatasreni illam, a famous center for astronomy and mathematics. He made a series of observations of the eclipses of the sun and the moon between 1395 and 1432 and composed several astronomical texts, the last of which was written in the 1450s,

[1] Newton (1959–1960) vol. 2, pp. 20–47, especially p. 36; Turnbull (1939) p. 170; Jyesthadeva et al. (2008).

near the end of his life. Sankara Variyar attributed to Paramesvara a formula for the radius of a circle in terms of the sides of an inscribed quadrilateral. See Exercise 4. Paramesvara's son, Damodara, was the teacher of Jyesthadeva (c. 1500–c. 1570) whose works survive and give us all the surviving proofs of this school. Damodara was also the teacher of Nilakantha (c. 1450–c. 1550) who composed the famous treatise called the *Tantrasangraha* (c. 1500), a digest of the mathematical and astronomical knowledge of his time. His works allow us determine his approximate dates because, in his *Aryabhatyabhasya*, Nilakantha refers to his observation of solar eclipses in 1467 and 1501. Nilakantha made several efforts to establish new parameters for the mean motions of the planets and vigorously defended the necessity of continually correcting astronomical parameters on the basis of observation. Sankara Variyar (c. 1500–1560) was his student.

The surviving texts containing results on infinite series are Nilakantha's *Tantrasangraha*, a commentary on it by Sankara Variyar called *Yuktidipika*, the *Yuktibhasa* by Jyesthadeva and the *Kriyakramakari*, started by Variyar and completed by his student Mahisamangalam Narayana. In addition, there is a text called *Karanapaddhati* of Putumana Somayaji, thought by some to have been written around 1700. However, the four translators of this work present an argument that Somayaji was a junior contemporary of Nilakantha and composed his work between 1532 and 1566. All these works are in Sanskrit except the *Yuktibhasa*, written in Malayalam, the language of Kerala. These works, especially the *Yuktibhasa* that gives detailed arguments, provide a summary of major results on series discovered by these original mathematicians of the indistinct past:

A. Series expansions for arctangent, sine, and cosine:

(1) $\theta = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \cdots,$

(2) $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots,$

(3) $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots,$

(4) $\sin^2\theta = \theta^2 - \frac{\theta^4}{2^2-\frac{2}{2}} + \frac{\theta^6}{(2^2-\frac{2}{2})(3^2-\frac{3}{2})} - \frac{\theta^8}{(2^2-\frac{2}{2})(3^2-\frac{3}{2})(4^2-\frac{4}{2})} + \cdots.$

In the proofs of the formulas contained in (A), the range of θ for the first series was $0 \le \theta \le \frac{\pi}{4}$ and for the second and third was $0 \le \theta \le \frac{\pi}{2}$. Although the series for sine and cosine converge for all real values, the concept of periodicity of the trigonometric functions was discovered much later.

B. Series for π:

(1) $\frac{\pi}{4} \approx 1 - \frac{1}{3} + \frac{1}{5} - \cdots \mp \frac{1}{n} \pm f_i(n+1), \quad i = 1, 2, 3,$ where

$$f_1(n) = \frac{1}{2n}, \quad f_2(n) = \frac{n}{2(n^2+1)},$$

and

$$f_3(n) = \frac{n^2+4}{2n(n^2+5)};$$

(2) $\frac{\pi}{4} = \frac{3}{4} + \frac{1}{3^3-3} - \frac{1}{5^3-5} + \frac{1}{7^3-7} - \cdots$;

(3) $\frac{\pi}{4} = \frac{4}{1^5+4\cdot 1} - \frac{4}{3^5+4\cdot 3} + \frac{4}{5^5+4\cdot 5} - \cdots$;

(4) $\frac{\pi}{2\sqrt{3}} = 1 - \frac{1}{3\cdot 3} + \frac{1}{5\cdot 3^2} - \frac{1}{7\cdot 3^3} + \cdots$;

(5) $\frac{\pi}{6} = \frac{1}{2} + \frac{1}{(2\cdot 2^2-1)^2-2^2} + \frac{1}{(2\cdot 4^2-1)^2-4^2} + \frac{1}{(2\cdot 6^2-1)^2-6^2} + \cdots$;

(6) $\frac{\pi-2}{4} \approx \frac{1}{2^2-1} - \frac{1}{4^2-1} + \frac{1}{6^2-1} - \cdots \mp \frac{1}{n^2-1} \pm \frac{1}{2((n+1)^2+2)}$;

(7) $\frac{\pi}{8} = \frac{1}{2^2-1} + \frac{1}{6^2-1} + \frac{1}{10^2-1} + \cdots$;

(8) $\frac{\pi}{8} = \frac{1}{2} - \frac{1}{4^2-1} - \frac{1}{8^2-1} - \frac{1}{12^2-1} - \cdots$.

These results were stated in verse form. Thus, the series for sine was described:[2]

> The arc is to be repeatedly multiplied by the square of itself and is to be divided [in order] by the square of each even number increased by itself and multiplied by the square of the radius. The arc and the terms obtained by these repeated operations are to be placed in sequence in a column, and any last term is to be subtracted from the next above, the remainder from the term then next above, and so on, to obtain the jya (sine) of the arc.

So if r is the radius and s the arc, then the successive terms of the repeated operations mentioned in the description are given by

$$s\cdot\frac{s^2}{(2^2+2)r^2},\quad s\cdot\frac{s^2}{(2^2+2)r^2}\cdot\frac{s^2}{(4^2+4)r^2},\ldots$$

and the equation is

$$y = s - s\cdot\frac{s^2}{(2^2+2)r^2} + s\cdot\frac{s^2}{(2^2+2)r^2}\cdot\frac{s^2}{(4^2+4)r^2} - \cdots,$$

where $y = r\ \sin\frac{s}{r}$.

Nilakantha's *Aryabhatyabhasya* attributes the sine series to Madhava. The *Kriyakramakari* attributes to Madhava the first two cases of (B.1), the arctangent series, and series (B.4); note that (B.4) can be derived from the arctangent by taking $\theta = \frac{\pi}{6}$. The extant manuscripts do not appear to attribute the other series to a particular person. The *Yuktidipika* gives series (B.6), including the remainder; it is possible that this series is due to Sankara Variyar, the author of the work. Series (B.7) and (B.8) are mentioned in the *Yuktibhasa* and are easily transformable into series (A.1) with $\theta = \frac{\pi}{4}$. We can safely conclude that the power series for arctangent, sine, and cosine were obtained by Madhava.

The series for $\sin^2\theta$, (A.4), follows directly from the series for $\cos\theta$ by an application of the double angle formula, $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$. The series for $\frac{\pi}{4}$, (B.1), has several points of interest. When $n\to\infty$, it is simply the series discovered by Leibniz in 1673, that he communicated to Newton.[3] However, this series is not useful for computational purposes because it converges extremely slowly. To make it more effective in this respect, Madhava added a rational approximation for the

[2] Rajagopal and Rangachari (1977) p. 96.
[3] Newton (1959–1960) vol II, pp. 57–71, especially p. 67.

remainder after n terms. We present Jyesthadeva's derivation for the expressions $f_1(n)$ and $f_2(n)$ in (B.1) later in this chapter. However, if we set

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots \mp \frac{1}{n} \pm f(n+1), \tag{1.1}$$

then the remainder $f(n)$ has the continued fraction expansion

$$f(n+1) = \frac{1}{2} \cdot \frac{1}{n+}\,\frac{1^2}{n+}\,\frac{2^2}{n+}\,\frac{3^2}{n+}\cdots, \tag{1.2}$$

where $f(n+1)$ satisfies the functional relation

$$f(n+1) + f(n-1) = \frac{1}{n}. \tag{1.3}$$

The first three convergents of this continued fraction are

$$\frac{1}{2n} = f_1(n+1), \quad \frac{n}{2(n^2+1)} = f_2(n+1), \quad \text{and} \quad \frac{1}{2}\,\frac{n^2+4}{n(n^2+5)} = f_3(n+1). \tag{1.4}$$

Although this continued fraction is not mentioned in any extant works of the Kerala school, their approximants indicate that they must have known it, at least implicitly. In fact, continued fractions appear in much earlier Indian works. The *Lilavati* of Bhaskara (c. 1150) used continued fractions to solve first-order Diophantine equations and Variyar's *Kriyakramakari* was a commentary on Bhaskara's book.

The approximation in equation (B.6) is similar to that in (B.1) and gives further evidence that the Kerala mathematicians saw a connection between series and continued fractions. If we write

$$\frac{\pi-2}{4} = \frac{1}{2^2-1} - \frac{1}{4^2-1} + \frac{1}{6^2-1} - \cdots \pm \frac{1}{n^2-1} \pm g(n+1), \tag{1.5}$$

then

$$g(n) = \frac{1}{2n} \cdot \frac{1}{n+}\,\frac{1\cdot 2}{n+}\,\frac{2\cdot 3}{n+}\,\frac{3\cdot 4}{n+}\cdots \tag{1.6}$$

and

$$g_1(n) = \frac{1}{2n}, \quad g_2(n) = \frac{1}{2(n^2+2)}. \tag{1.7}$$

Newton, who was very interested in the numerical aspects of series, also found the $f_1(n) = \frac{1}{2n}$ approximation when he saw Leibniz's series. He wrote in a letter in 1676[4] to Henry Oldenburg:

> By the series of Leibniz also if half the term in the last place be added and some other like device be employed, the computation can be carried to many figures.

[4] Newton (1959–1960) vol. 2, pp. 110–149, especially p. 140.

Though the accomplishments of Madhava and his followers are quite impressive, the members of the school do not appear to have had any interaction with people outside of the very small region where they lived and worked. By the end of the sixteenth century, the school ceased to produce any further original works. Thus, there appears to be no continuity between the ideas of the Kerala scholars and those outside India or even from other parts of India.

1.2 Transformation of Series

The series in equations (B.2) and (B.3) are transformations of

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$$

by means of the rational approximations for the remainder. To understand this transformation in modern notation, observe:

$$\frac{\pi}{4} = (1 - f_1(2)) - \left(\frac{1}{3} - f_1(2) - f_1(4)\right) + \left(\frac{1}{5} - f_1(4) - f_1(6)\right) - \cdots . \quad (1.8)$$

The $(n+1)$th term in this series is

$$\frac{1}{2n+1} - f_1(2n) - f_1(2n+2) = \frac{1}{2n+1} - \frac{1}{4n} - \frac{1}{4(n+1)} = \frac{-1}{(2n+1)^3 - (2n+1)}. \quad (1.9)$$

Thus, we arrive at equation (B.2). Equation (B.3) is similarly obtained:

$$\frac{\pi}{4} = (1 - f_2(2)) - \left(\frac{1}{3} - f_2(2) - f_2(4)\right) + \left(\frac{1}{5} - f_2(4) - f_2(6)\right) - \cdots , \quad (1.10)$$

and here the $(n+1)$th term is

$$\frac{1}{2n+1} - \frac{n}{(2n)^2+1} - \frac{n+1}{(2n+2)^2+1} = \frac{4}{(2n+1)^5 + 4(2n+1)}. \quad (1.11)$$

Clearly, the nth partial sums of these two transformed series can be written as

$$s_i(n) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \mp \frac{1}{2n-1} \pm f_i(2n), \quad i = 1, 2. \quad (1.12)$$

Since series (1.8) and (1.10) are alternating, and the absolute values of the terms are decreasing, it follows that

$$\frac{1}{(2n+1)^3 - (2n+1)} - \frac{1}{(2n+3)^3 - (2n+3)} < \left|\frac{\pi}{4} - s_1(n)\right| < \frac{1}{(2n+1)^3 - (2n+1)}. \quad (1.13)$$

Also

$$\frac{4}{(2n+1)^5+4(2n+1)} - \frac{4}{(2n+3)^5+4(2n+3)} < \left|\frac{\pi}{4} - s_2(n)\right| < \frac{4}{(2n+1)^5+4(2n+1)}. \tag{1.14}$$

Thus, taking fifty terms of $1 - \frac{1}{3} + \frac{1}{5} - \cdots$ and using the approximation $f_2(n)$, the last inequality shows that the error in the value of π becomes less than 4×10^{-10}. The Leibniz series with fifty terms is normally accurate in computing π up to only one decimal place; by contrast, the Keralese method of rational approximation of the remainder produces numerically useful results.

1.3 Jyesthadeva on Sums of Powers

The Sanskrit texts of the Kerala school with few exceptions contain merely the statements of results without derivations. It is therefore extremely fortunate that Jyesthadeva's Malayalam text *Yuktibhasa*, containing the methods for obtaining the formulas, has survived. Sankara Variyar's *Yuktidipika* is a modified Sanskrit version of the *Yuktibhasa*. It seems that the *Yuktibhasa* was the text used by Jyesthadeva's students at his illam. From this, one may surmise that Variyar, a student of Nilakantha, also studied with Jyesthadeva whose illam was very close to that of Nilakantha.

A basic result used by the Kerala school in the derivation of their series is that

$$\lim_{n\to\infty} \frac{1}{n^{k+1}} \sum_{j=1}^{n} j^k = \frac{1}{k+1}. \tag{1.15}$$

Jyesthadeva gave an inductive proof of this result.[5] He noted

$$\begin{aligned} S_n^{(1)} &= n + (n-1) + (n-2) + \cdots + 1 \\ &= n \cdot n - \big(1 + 2 + \cdots + (n-1)\big) \\ &= n^2 - S_{n-1}^{(1)}. \end{aligned}$$

He then observed that for large n, $S_{n-1}^{(1)} \approx S_n^{(1)}$ and hence

$$S_n^{(1)} \approx n^2 - S_n^{(1)} \quad \text{or} \quad S_n^{(1)} \approx \frac{1}{2} n^2. \tag{1.16}$$

Now

$$S_n^{(2)} = n^2 + (n-1)^2 + \cdots + 1^2$$

[5] Jyesthadeva et al. (2008) pp. 192–196.

and

$$nS_n^{(1)} = n\big(n + (n-1) + \cdots + 1\big),$$

so that

$$\begin{aligned} nS_n^{(1)} - S_n^{(2)} &= 1 \cdot (n-1) + 2(n-2) + 3(n-3) + \cdots + (n-1) \cdot 1 \\ &= (n-1) + (n-2) + (n-3) + \cdots + 1 \\ &\qquad + (n-2) + (n-3) + \cdots + 1 \\ &\qquad\qquad + (n-3) + \cdots + 1 \\ &\qquad\qquad\qquad \cdots\cdots\cdots \\ &= S_{n-1}^{(1)} + S_{n-2}^{(1)} + S_{n-3}^{(1)} + \cdots + S_1^{(1)}. \end{aligned} \tag{1.17}$$

By using (1.16) in (1.17), Jyesthadeva had, for large n,

$$\begin{aligned} nS_n^{(1)} - S_n^{(2)} &\approx \frac{1}{2}(n-1)^2 + \frac{1}{2}(n-2)^2 + \frac{1}{2}(n-3)^2 + \cdots + \frac{1}{2} \cdot 1^2 \\ &\approx \frac{1}{2} S_n^{(2)}. \end{aligned} \tag{1.18}$$

Note that for large n, $S_n^{(2)} \approx S_{n-1}^{(2)}$ was used to obtain (1.18). Again, by applying (1.16) in (1.18), he had

$$S_n^{(2)} \approx \frac{1}{3} n^3. \tag{1.19}$$

Next

$$\begin{aligned} nS_n^{(2)} - S_n^{(3)} &= 1 \cdot (n-1)^2 + 2 \cdot (n-2)^2 + \cdots + (n-1) \cdot 1^2 \\ &= S_{n-1}^{(2)} + S_{n-2}^{(2)} + \cdots + S_1^{(2)}. \end{aligned}$$

Applying equation (1.19) yielded

$$\begin{aligned} nS_n^{(2)} - S_n^{(3)} &\approx \frac{1}{3}(n-1)^3 + \cdots + \frac{1}{3} \cdot 1^3 \\ &\approx \frac{1}{3} S_n^{(3)} \end{aligned}$$

or

$$S_n^{(3)} \approx \frac{1}{4} n^4.$$

Jyesthadeva next presented the general principle of summation, that we may express within our notation as:

Let

$$S_n^{(k)} = n^k + (n-1)^k + \cdots + 1^k$$

and suppose that $S_n^{(k-1)}$ has been estimated to be

$$S_n^{(k-1)} \approx \frac{1}{k} n^k.$$

Then

$$\begin{aligned} nS_n^{(k-1)} - S_n^{(k)} &= S_{n-1}^{(k-1)} + S_{n-2}^{(k-1)} + \cdots + S_1^{(k-1)} \\ &\approx \frac{1}{k}\big((n-1)^k + (n-2)^k + \cdots + 1^k\big) \\ &\approx \frac{1}{k} S_n^{(k)}; \end{aligned} \tag{1.20}$$

hence

$$S_n^{(k)} \approx \frac{1}{k+1} n^{k+1} \tag{1.21}$$

and this proved (1.15).

As noted, Jyesthadeva did not write formulas in the symbolic form we have used. Rather, he gave verbal descriptions of his relations and formulas. His application of induction is very clearly executed. He writes that the case $k = 1$ implies the case $k = 2$, that in turn implies the case $k = 3$; in the same manner, a higher value of k will imply the next value, and so on.[6] Also note that formula (1.20) was known to al-Haytham (965–1039) for $k = 1$ to $k = 4$, but who most probably knew that it could be generalized, though he did not do it explicitly.[7]

1.4 Arctangent Series in the *Yuktibhasa*

The derivation of the arctangent series,[8] as given by Jyesthadeva, boils down to the integration of $\frac{1}{1+x^2}$, as do the methods of Gregory and Leibniz.

In Figure 1.1, AC is a quarter circle of radius one with center O; $OABC$ is a square. The side AB is divided into n equal parts of length δ so that $n\delta = 1$ and $P_{k-1}P_k = \delta$. EF and $P_{k-1}D$ are perpendicular to OP_k. Now, the triangles OEF and $OP_{k-1}D$ are similar, implying that

$$\frac{EF}{OE} = \frac{P_{k-1}D}{OP_{k-1}} \quad \text{or} \quad EF = \frac{P_{k-1}D}{OP_{k-1}}.$$

The similarity of the triangles $P_{k-1}P_kD$ and OAP_k gives

$$\frac{P_{k-1}P_k}{OP_k} = \frac{P_{k-1}D}{OA} \quad \text{or} \quad P_{k-1}D = \frac{P_{k-1}P_k}{OP_k}.$$

[6] ibid. pp. 65–66.
[7] See Katz (1995) p. 125.
[8] Jyesthadeva et al. (2008) pp. 183–191.

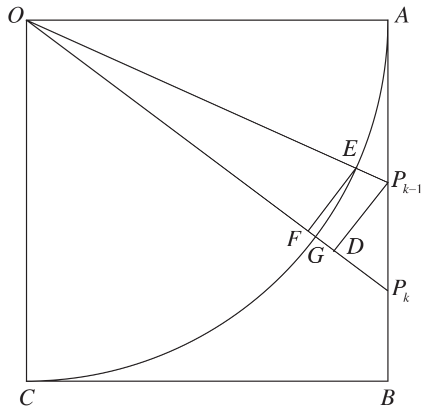

Figure 1.1 Rectifying a circle by the arctangent series.

Thus,

$$EF = \frac{P_{k-1}P_k}{OP_{k-1}OP_k} \simeq \frac{P_{k-1}P_k}{OP_k^2} = \frac{P_{k-1}P_k}{1+AP_k^2} = \frac{\delta}{1+k^2\delta^2}.$$

Now

$$\text{arc } EG \simeq EF \simeq \frac{\delta}{1+k^2\delta^2},$$

and if we write $AP_k = x = \tan\theta$, where $\theta = A\widehat{O}P_k$, then

$$\arctan x = \lim_{k\to\infty} \sum_{j=1}^{k} \frac{\delta}{1+j^2\delta^2}. \tag{1.22}$$

To compute this limit, Jyesthadeva expanded $\frac{1}{1+j^2\delta^2}$ as a geometric series. He derived the series by an iterative procedure:

$$\frac{1}{1+x} = 1 - x\left(\frac{1}{1+x}\right) = 1 - x\left(1 - x\left(\frac{1}{1+x}\right)\right).$$

Thus, (1.22) is converted to

$$\begin{aligned}\arctan x &= \lim_{k\to\infty}\left(\delta\sum_{j=1}^{k}1 - \delta^3\sum_{j=1}^{k}j^2 + \delta^5\sum_{j=1}^{k}j^4 - \cdots\right)\\ &= \lim_{k\to\infty}\left(\frac{x}{k}\sum_{j=1}^{k}1 - \frac{x^3}{k^3}\sum_{j=1}^{k}j^2 + \frac{x^5}{k^5}\sum_{j=1}^{k}j^4 - \cdots\right)\\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots.\end{aligned}$$

The last step follows from (1.15). Note that this is the Madhava–Gregory series for $\arctan x$ and the series for $\frac{\pi}{4}$ follows by taking $x = 1$.

1.5 Derivation of the Sine Series in the *Yuktibhasa*

Once again, Jyesthadeva's derivation of the sine series has similarities with Leibniz's derivation of the cosine series. In Figure 1.2, suppose that $A\widehat{O}P = \theta$, $OP = R$, P is the midpoint of the arc $P_{-1}P_1$, and PQ is perpendicular to OA, where O is the origin of the coordinate system. Let $P = (x, y)$, $P_1 = (x_1, y_1)$, and $P_{-1} = (x_{-1}, y_{-1})$. From the similarity of the triangles $P_{-1}Q_1P_1$ and OPQ, we have

$$\frac{P_{-1}P_1}{OP} = \frac{x_{-1} - x_1}{y} = \frac{y_1 - y_{-1}}{x}. \tag{1.23}$$

Jyesthadeva took an arc, $P_{-1}P = R\frac{\Delta\theta}{2} = \frac{\Delta s}{2}$, small enough that he could set it equal to the line segment $P_{-1}P$; we can then write (1.23) as

$$\cos\left(\theta + \frac{\Delta\theta}{2}\right) - \cos\left(\theta - \frac{\Delta\theta}{2}\right) = -\sin\theta\,\Delta\theta \tag{1.24}$$

and

$$\sin\left(\theta + \frac{\Delta\theta}{2}\right) - \sin\left(\theta - \frac{\Delta\theta}{2}\right) = \cos\theta\,\Delta\theta. \tag{1.25}$$

In fact, in his *Siddhanta Siromani*, Bhaskara[9] had stated (1.25) and proved it in the same way; he applied it to the discussion of the instantaneous motion of planets. Interestingly, in the 1650s, Pascal[10] used a very similar argument to show that $\int \cos\theta\, d\theta = \sin\theta$ and $\int \sin\theta\, d\theta = -\cos\theta$.

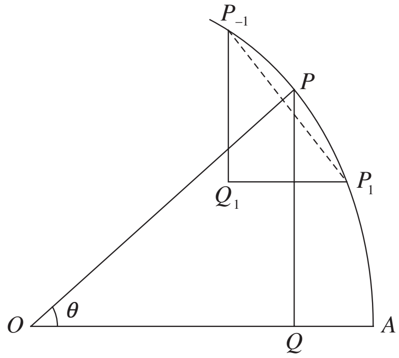

Figure 1.2 Derivation of the sine series.

[9] Bhaskara (2010).
[10] Struik (1969) vol. 2, p. 239.