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q-Series

25.1 Preliminary Remarks

The theory of q-series in modern mathematics plays a significant role in partition theory and modular functions as well as in some aspects of Lie algebras and statistical mechanics. This subject began quietly, however, with two combinatorial problems posed in a September 1740 letter from Phillipe Naudé (1684–1747) to Euler. Naudé was a mathematician of French origin working in Berlin. In general, his question was how to find the number of ways in which a given number could be expressed as the sum of a fixed number, first of distinct integers and then without the requirement that the integers in the sum be distinct. For example, in how many ways can 50 be expressed as a sum of 7 distinct/not necessarily distinct integers?[1]

As an example of both these problems, 7 can be expressed as a sum of three distinct integers in one way, $1+2+4$; whereas it can be expressed as a sum of three integers in four ways: $1+1+5$, $1+2+4$, $1+3+3$, $2+2+3$. Euler received Naudé's letter in St. Petersburg, just before he moved to Berlin. Within two weeks, in a reply to Naudé, Euler outlined a solution and soon after that he presented his complete solution to the Petersburg Academy.[2] In 1748, he devoted a whole chapter to this topic in his *Introductio in Analysin Infinitorum*.[3] The essential idea in Euler's solution was that the coefficient of $q^k x^m$ in the series expansion of the infinite product

$$f(q,x) = (1+qx)(1+q^2x)(1+q^3x)\cdots \qquad (25.1)$$

gave the number of ways of writing k as a sum of m distinct positive integers. Euler used the functional relation

$$f(q,x) = (1+qx)\,f(q,qx) \qquad (25.2)$$

[1] See Eu. 1-2 pp. 163–193, especially § 19, E 158 § 19 and Weil (1983) pp. 276–277.
[2] Eu. 1-2 pp. 163–193, E 158.
[3] Euler (1988) chapter 16, especially pp. 256–270.

to prove that

$$f(q,x) = \sum_{m=0}^{\infty} \frac{q^{\frac{m(m+1)}{2}} x^m}{(1-q)(1-q^2)\cdots(1-q^m)}. \tag{25.3}$$

He noted that

$$\begin{aligned} &\frac{1}{(1-q)(1-q^2)\cdots(1-q^m)} \\ &= (1+q+q^{1+1}+\cdots)(1+q^2+q^{2+2}+\cdots)\cdots(1+q^m+q^{m+m}+\cdots) \\ &= \sum_{n=0}^{\infty} a_n q^n, \end{aligned} \tag{25.4}$$

where the middle product showed that a_n, the coefficient of q^n, was the number of ways of writing n as a sum of integers chosen from the set $1, 2, \ldots, m$. This implied that the coefficient of $q^k x^m$ on the right-hand side of (25.3) was the number of ways of writing $k - \frac{m(m+1)}{2}$ as a sum of integers from the set $1, 2, \ldots, m$. Thus, Euler stated the theorem: The number of different ways in which the number n can be expressed as a sum of m different numbers is the same as the number of different ways in which $n - \frac{m(m+1)}{2}$ can be expressed as the sums of the numbers $1, 2, 3, \ldots, m$.

For the second problem, Euler used the product

$$g(q,x) = \prod_{n=1}^{\infty} (1-q^n x)^{-1} \tag{25.5}$$

and obtained the corresponding series and theorem in a similar way. Euler here used functional relations to evaluate the product as a series, just as he earlier employed functional relations to evaluate the beta integral as a product. Of course, this method goes back to Wallis.

Euler also considered the case $x = 1$. In that case (in modern notation), we have

$$\begin{aligned} \prod_{n=1}^{\infty}(1-q^n)^{-1} &= (1+q+q^{1+1}+\cdots)(1+q^2+q^{2+2}+\cdots)(1+q^3+q^{3+3}+\cdots)\cdots \\ &= \sum_{n=0}^{\infty} p(n) q^n, \end{aligned} \tag{25.6}$$

where $p(n)$ is the number of partitions of n, or the number of ways in which n can be written as a sum of positive integers. For example, $p(4) = 5$ because 4 has the five partitions

$$1+1+1+1,\ 2+1+1,\ 2+2,\ 3+1,\ 4.$$

The product in (25.6) also led Euler to consider its reciprocal, $\prod_{n=1}^{\infty}(1-q^n)$. He attempted to expand this as a series but it took him nine years to completely resolve this difficult problem. In his first attempt presented in 1741 but published ten years later,[4] he multiplied a large number of terms of the product to find that

$$\prod_{n=1}^{\infty}(1-q^n) = 1-q-q^2+q^5+q^7-q^{12}-q^{15}+q^{22}+q^{26}-q^{35}-q^{40}+q^{51}+\cdots. \tag{25.7}$$

He quickly found a general expression for the exponents, $\frac{m(3m\pm1)}{2}$. He most probably did this by considering the differences in the sequence of exponents; note that the sequence of exponents is

$$0, 1, 2, 5, 7, 12, 15, 22, 26, 35, 40, 51, \ldots.$$

Observe that the sequence of differences is then

$$1, 1, 3, 2, 5, 3, 7, 4, 9, 5, 11, \ldots.$$

The pattern of this sequence suggests that one should group the sequence of exponents into two separate sequences, first taking the exponents of the odd-numbered terms and then the exponents of the even-numbered terms.[5] For example, the sequence of exponents of the odd-numbered terms is 0, 2, 7, 15, 26, 40, ..., and their differences are 2, 5, 8, 11, 14, ... Since the differences of these differences are 3 in every case, we may apply the formula of Zhu Shijie and Montmort, given in Section 10.3, to perceive that the $(n+1)$th term of the sequence of odd-numbered exponents will be given by

$$0+2n+\frac{3n(n-1)}{2} = \frac{n(3n+1)}{2}.$$

Similarly, the nth term in the sequence of even-term exponents is $\frac{n(3n-1)}{2}$. In the *Introductio*, Euler wrote, "If we consider this sequence with some attention we will note that the only exponents which appear are of the form $\frac{(3n^2\pm n)}{2}$ and that the sign of the corresponding term is negative when n is odd, and the sign is positive when n is even."[6] Thus, Euler made the conjecture

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{m=-\infty}^{\infty}(-1)^m q^{\frac{m(3m+1)}{2}} = 1+\sum_{m=1}^{\infty}(-1)^m q^{\frac{m(3m\pm1)}{2}}, \tag{25.8}$$

and finally found a proof of this in 1750. He immediately wrote Goldbach about the details of the proof,[7] explaining that it depended on the algebraic identity:

$$\begin{aligned}&(1-\alpha)(1-\beta)(1-\delta)\text{ etc.}\\ &\quad = 1-\alpha-\beta(1-\alpha)-\gamma(1-\alpha)(1-\beta)-\delta(1-\alpha)(1-\beta)(1-\gamma)-\text{etc.}\end{aligned}$$

[4] Eu. 1-2 pp. 163–193, especially p. 193. E 158 § 37.
[5] Eu. 1-2 pp. 241–253. E 175 § 8.
[6] Euler (1988) p. 274.
[7] Fuss (1968) vol. 1, pp. 522–524. See also Eu. 1-2 pp. 390–398. E 244.

This identity is easy to check, since the first three terms on the right-hand side add up to

$$1 - \alpha - \beta(1-\alpha) = (1-\alpha)(1-\beta),$$

and when this is added to the fourth term, we get

$$(1-\alpha)(1-\beta) - \gamma(1-\alpha)(1-\beta) = (1-\alpha)(1-\beta)(1-\gamma),$$

and so on. An interesting feature of the series (25.8) is that the exponent of q is a quadratic in m, the index of summation. Surprisingly, series of this kind had already appeared in 1690 within Jakob Bernoulli's works on probability theory,[8] but he was unable to do much with them. Over a century later, Gauss initiated a systematic study of these series. Entry 58 of Gauss's mathematical diary, dated February 1797, gives a continued fraction expansion of one of Bernoulli's series:[9]

$$1 - a + a^3 - a^6 + a^{10} - \cdots = \cfrac{1}{1+\cfrac{a}{1+\cfrac{a^2-a}{1+\cfrac{a^3}{1+\cfrac{a^4-a^2}{1+\cfrac{a^5}{1+\text{etc.}}}}}}} \tag{25.9}$$

In his diary, Gauss added the comment, "From this all series where the exponents form a series of the second order are easily transformed." About a year later, he raised the problem of expressing $1 + q + q^3 + q^6 + q^{10} + \cdots$ as an infinite product. Gauss came upon series of this type around 1794 in the context of his work on the arithmetic-geometric mean, that he had been studying since 1791.[10] This latter work was absorbed into his theory of elliptic functions. Series (25.8) and (25.9) are actually examples of the special kind of q-series called theta functions. Theta functions also arose naturally in Fourier's 1807 study of heat conduction.

Unfortunately, Gauss did not publish any of his work on theta or elliptic functions, and it remained for Abel and Jacobi to independently rediscover much of this work, going beyond Gauss in many respects. Around 1805–1808, Gauss began to view q-series in a different way. For example, his 1808 paper[11] on q-series dealt with a generalization of the binomial coefficient and the binomial series. In particular, he defined the Gaussian polynomial

[8] Bernoulli and Sylla (2006) pp. 176–180.
[9] See Dunnington (2004) p. 474.
[10] Peters (1860–1865) vol. 1, p. 125. See also Gauss (1863–1927) vol. 3, pp. 361–371, also vol. 10, part 2, p. 18 of Schlesinger's article on Gauss's work in function theory.
[11] Gauss (1981) pp. 463–495.

$$(m, \mu) = \frac{(1-q^m)(1-q^{m-1})(1-q^{m-2})\cdots(1-q^{m-\mu+1})}{(1-q)(1-q^2)(1-q^3)\cdots(1-q^\mu)}. \tag{25.10}$$

Note that Gauss wrote x instead of q. Observe that as $q \to 1$

$$(m, \mu) \to \binom{m}{\mu}. \tag{25.11}$$

This work led to an unexpected byproduct: an evaluation of the Gauss sum $\sum_{k=0}^{n-1} e^{\frac{2\pi i k^2}{n}}$ where n was an odd positive integer. This sum had already appeared naturally in Gauss's theory of the cyclotomic equation $x^n - 1 = 0$, to which he had devoted the final chapter of his 1801 *Disquisitiones Arithmeticae*.[12] There Gauss had computed the square of the Gauss sum, but he was unable to determine the correct sign of the square root. Already in 1801, he knew that it was important to find the exact value of the sum; he expended considerable effort over the next four years to compute the Gauss sum, and it was a complete surprise for him when the result dropped out of his work on q-series. In September 1805, he wrote his astronomer friend, Wilhelm Olbers,[13]

> What I wrote there [Disqu. Arith. section 365] ..., I proved rigorously, but I was always annoyed by what was missing, namely, the determination of the sign of the root. This gap spoiled whatever else I found, and hardly a week may have gone by in the last four years without one or more unsuccessful attempts to unravel this knot - just recently it again occupied me much. But all the brooding, the searching, was to no avail, and I had sadly to lay down my pen again. A few days ago, I finally succeeded - not by my efforts, but by the grace of God, I should say. The mystery was solved the way lightning strikes, I myself could not find the connection between what I knew previously, what I investigated last, and the way it was finally solved.

He recorded these events in his diary:[14]

> (May 1801) A method for proving the first fundamental theorem has been found by means of a most elegant theorem in the division of the circle, thus
>
> $$\sum \frac{\sin}{\cos} \frac{nn}{a} P = + \begin{array}{c} \sqrt{a} \\ \sqrt{a} \end{array} \Bigg| \begin{array}{c} 0 \\ +\sqrt{a} \end{array} \Bigg| \begin{array}{c} 0 \\ 0 \end{array} \Bigg| \begin{array}{c} +\sqrt{a} \\ 0 \end{array} \tag{25.12}$$
>
> according as $a \equiv 0, 1, 2, 3 \pmod 4$ substituting for n all numbers from 0 to $(a-1)$. (August 1805) The proof of the most charming theorem recorded above, May 1801, which we have sought to prove for 4 years and more with every effort, at last perfected.

Conceptually, this was a major achievement, since it served to connect cyclotomy with the reciprocity law. Gauss may have initially considered the polynomial $\sum_{k=0}^{m}(m,k)x^k$ as a possible analog of the finite binomial series. In any case, he expressed the sum as a finite product when $x = -1$ and when $x = \sqrt{q}$, and these formulas finally yielded the correct value of the Gauss sum. It is interesting to note that the polynomial

12 For an interesting commentary on Gauss's work in cyclotomy, see Neumann (2007a)) and (2007b).
13 See Bühler (1981) p. 31.
14 Dunnington (2004) p. 481.

$\sum_{k=0}^{m}(m,k)x^k$ played a key role in Szegő's theory of orthogonal polynomials on the unit disc.

Gauss found the appropriate q-extension of the terminating binomial theorem, perhaps around 1808, but he did not publish it. In 1811, Heinrich A. Rothe (1773–1841) first published this result in the preface of his *Systematisches Lehrbuch der Arithmetik*[15] as the formula

$$\sum_{k=0}^{m} \frac{1-q^m}{1-q} \cdot \frac{1-q^{m-1}}{1-q^2} \cdot \dots \cdot \frac{1-q^{m-k+1}}{1-q^k} \cdot q^{\frac{k(k+1)}{2}} x^{m-k} y^k$$
$$= (x+y)(x+qy)\cdots(x+q^{k-1}y). \tag{25.13}$$

Although this was the most important result in the book, Rothe excluded it from the body of text, apparently in order to keep the book within the size required by the publisher. Gauss's paper and Rothe's formula indicated a direction for further research on q-series relating to the extension of the binomial theorem. This path was not pursued until the 1840s, except in Schweins's *Analysis* of 1820.[16] This work presented a q-extension of Vandermonde's identity (25.64).

In the 1820s, Jacobi investigated q-series in connection with his work on theta functions, a byproduct of his researches on elliptic functions. His most remarkable discovery in this area was the triple product identity. Jacobi's famous *Fundamenta Nova* of 1829 stated the formula as[17]

$$(1+qz)(1+q^3z)(1+q^5z)\cdots\left(1+\frac{q}{z}\right)\left(1+\frac{q^3}{z}\right)\left(1+\frac{q^5}{z}\right)\cdots$$
$$= \frac{1+q\left(z+\frac{1}{z}\right)+q^4\left(z^2+\frac{1}{z^2}\right)+q^9\left(z^3+\frac{1}{z^3}\right)+\cdots}{(1-q^2)(1-q^4)(1-q^6)(1-q^8)\cdots}. \tag{25.14}$$

Jacobi regarded this identity as his most important formula in pure mathematics. He gave several very important applications. In one of these, he derived an identity, giving the number of representations of an integer as a sum of four squares. In another, he obtained an important series expression for the square root of the period of some elliptic functions, allowing him to find a new derivation of the following transformation of a theta function, originally due to Cauchy and Poisson:

$$1+2\sum_{n=1}^{\infty} e^{-n^2\pi x} = \frac{1}{\sqrt{x}}\left(1+2\sum_{n=1}^{\infty} e^{-\frac{n^2\pi}{x}}\right). \tag{25.15}$$

Jacobi also published a long paper on those series whose powers are quadratic forms; the triple product identity formed the basis for this. In the 1820s, when Gauss

15 Rothe (1811).
16 Schweins (1820) pp. 292–293.
17 Jacobi (1969) vol. 1, p. 234.

learned of Jacobi's work, he informed Jacobi that he had already found (25.14) in 1808. Legendre, on very friendly terms with Jacobi, refused to believe that Gauss had anticipated his friend. In a letter to Jacobi,[18] Legendre wrote, "Such outrageous impudence is incredible in a man with enough ability of his own that he should not have to take credit for other people's discoveries."[19] Then again, Legendre had had his own priority disputes with Gauss with regard to quadratic reciprocity and the method of least squares.

In the early 1840s, papers on q-series appeared in quick succession by Cauchy in France and Eisenstein, Jacobi, and E. Heine in Germany. As a second-year student at Berlin in 1844, Eisenstein presented twenty-five papers for publication to *Crelle's Journal*. One of them, "Neuer Beweis und Verallgemeinerung des binomischen Lehrsatzes,"[20] It began with the statement and proof of the Rothe–Gauss theorem; it then applied Euler's approach to the proof of the binomial theorem to obtain a version of the q-binomial theorem. Some details omitted by Euler in his account were treated in Eisenstein's paper.

Jacobi and Cauchy stated and proved the q-binomial theorem in the form

$$1+\frac{v-w}{1-q}z+\frac{(v-w)(v-qw)}{(1-q)(1-q^2)}z^2+\frac{(v-w)(v-qw)(v-q^2w)}{(1-q)(1-q^2)(1-q^3)}z^3+\cdots$$
$$=\frac{(1-wz)(1-qwz)(1-q^2wz)(1-q^3wz)\cdots}{(1-vz)(1-qvz)(1-q^2vz)(1-q^3vz)\cdots}. \qquad (25.16)$$

The idea in this proof was the same as the one used by Euler to prove (25.3), clearly a particular case. Jacobi also went on to give a q-extension of Gauss's ${}_2F_1$ summation formula. At that time, it was natural for someone to consider the q-extension of a general ${}_2F_1$ hypergeometric series; E. Heine did just that, and we discuss his work in Chapter 27.

25.2 Jakob Bernoulli's Theta Series

It is interesting that the series with quadratic exponents, normally arising in the theory of elliptic functions, occurred in Bernoulli's work in probability. In 1685, he proposed the following two problems in the *Journal des Savans*:

> Let there be two players A and B, playing against each other with two dice on the condition that whoever first throws a 7 will win. There are sought their expectations if they play in one of these orders:
>
> (1) A once, B once, A twice, B twice, A three times, B three times, A four times, B four times, etc.
>
> (2) A once, B twice, A three times, B four times, A five times, etc.

[18] Jacobi (1969) vol. 1, pp. 396–399, especially p. 398.
[19] For the English translation, see Remmert (1998) p. 29.
[20] Eisenstein (1975) vol. 1, pp. 117–121.

In his *Ars Conjectandi*, Bernoulli gave a solution, saying that in May 1690, when no solution to this problem had yet appeared, he communicated a solution to *Acta Eruditorum*.[21] In the first case, Bernoulli gave the probability for A to win as

$$1 - m + m^2 - m^4 + m^6 - m^9 + m^{12} - m^{16} + m^{20} - m^{25} + \text{ etc.} \tag{25.17}$$

In the second case, the probability for A to win was

$$1 - m + m^3 - m^6 + m^{10} - m^{15} + m^{21} - m^{28} + m^{36} - m^{45} + \text{ etc.} \tag{25.18}$$

In both cases, $m = \frac{5}{6}$. To make the quadratic exponents explicit, write the two series as

$$1 + \sum_{n=1}^{\infty} m^{n(n+1)} - \sum_{n=1}^{\infty} m^{n^2} \quad \text{and} \quad \sum_{n=0}^{\infty} (-1)^n m^{\frac{n(n+1)}{2}}.$$

Bernoulli remarked that the summation of these series was difficult because of the unequal jumps in the powers of m. He noted that numerical approximation to any degree of accuracy was easy and for $m = \frac{5}{6}$, the value of the second series was 0.52393; we remark that this value is inaccurate by only one in the last decimal place. Jakob Bernoulli was very interested in polygonal and figurate numbers; in fact, he worked out the sum of the reciprocals of triangular numbers. Here he had series with triangular and square numbers as exponents. Gauss discovered a way to express these series as products. Euler found the product expansion of a series with pentagonal numbers as exponents.

25.3 Euler's q-Series Identities

In response to the problems of Naudé, Euler proved the two identities:

$$(1 + qx)(1 + q^2x)(1 + q^3x)(1 + q^4x)\cdots$$
$$= 1 + \frac{q}{1-q}x + \frac{q^3}{(1-q)(1-q^2)}x^2 + \cdots + \frac{q^{\frac{m(m+1}{2}}}{(1-q)\cdots(1-q^m)}x^m + \cdots, \tag{25.19}$$

$$\frac{1}{(1-qx)(1-q^2x)(1-q^3x)\cdots}$$
$$= 1 + \frac{q}{1-q}x + \frac{q^2}{(1-q)(1-q^2)}x^2 + \cdots + \frac{q^m}{(1-q)\cdots(1-q^m)}x^m + \cdots. \tag{25.20}$$

Euler's argument for the first identity was outlined in the opening remarks of this chapter. His proof of his second identity ran along similar lines. We here follow Euler's

[21] Bernoulli and Sylla (2006) pp. 176–180.

presentation from his *Introductio*,[22] noting that Euler wrote x for our q and z for our x. Note also that the term q-series came into use only in the latter half of the nineteenth century, appearing in the works of Cayley, Rogers, and others. Jacobi may possibly have been the first to use the symbol q in the context of elliptic functions, though he did not use the term q-series. Euler let Z denote the infinite product on the left of (25.20) and he assumed that Z could be expanded as a series:

$$Z = 1 + Px + Qx^2 + Rx^3 + Sx^4 + \cdots . \tag{25.21}$$

When x was replaced by qx in Z, he got

$$\frac{1}{(1-q^2x)(1-q^3x)(1-q^4x)\cdots} = (1-qx)Z.$$

Making the same substitution in (25.21), he obtained

$$(1-qx)Z = 1 + Pqx + Qq^2x^2 + Rq^3x^3 + Sq^4x^4 + \cdots . \tag{25.22}$$

When the series for Z was substituted in (25.22) and the coefficients of the various powers of x were equated, the result was

$$P = \frac{q}{1-q},\; Q = \frac{Pq}{1-q^2},\; R = \frac{Qq}{1-q^3},\; S = \frac{Rq}{1-q^4}, \text{ etc.}$$

and this proved (25.20).

25.4 Euler's Pentagonal Number Theorem

Pentagonal numbers can be generated by the exponents $\frac{m(3m\pm1)}{2}$ in Euler's formula (25.8)

$$\prod_{n=1}^{\infty}(1-x^n) = 1 + \sum_{m=1}^{\infty}(-1)^m x^{\frac{m(3m\pm1)}{2}}. \tag{25.23}$$

This identity is often referred to as the pentagonal number theorem.

Recall that Euler had conjectured this result in 1741; he was convinced that this formula was valid, but he could not prove it. He was so confident of his conjecture that in 1747, he used this formula to prove a remarkable theorem on the sum of the divisors of an integer.[23] Concerning this theorem, he remarked in section 25.9 of his paper, "Indeed, I have no other proof."

To understand the 1747 theorem in which he used his conjecture, let n be a nonzero integer and let $\sigma(n) = \sum_{d|n} d$. Observe that if n were a negative integer,

[22] Euler (1988) pp. 361–363.
[23] See Euler's letter to Goldbach: Fuss (1968) vol. I, pp. 407–408. Also see his paper E 175.

then $\sigma(n) = 0$. We remark that Euler's own notation for $\sigma(n)$ was $\int n$. Thus, for n a positive integer, Euler's formula was

$$\sigma(n) = \sigma(n-1) + \sigma(n-2) - \sigma(n-5) - \sigma(n-7) + \cdots, \tag{25.24}$$

where if $n = \frac{m(3m\mp1)}{2}$, then $\sigma(0) = n$. Here the numbers $1, 2, 5, 7 \ldots$ in (25.24) are the pentagonal numbers. In his proof, here given in modernized and more brief notation, Euler first took the logarithmic derivative of

$$\sum_{n=0}^{\infty} (-1)^n x^{\frac{n(3n\mp1)}{2}} = \prod_{m=1}^{\infty} (1 - x^m) \tag{25.25}$$

to obtain

$$\sum_{n=1}^{\infty} (-1)^n \frac{n(3n\mp1)}{2} x^{\frac{n(3n\mp1)}{2}} = -\left(\sum_{n=0}^{\infty} (-1)^n x^{\frac{n(3n\mp1)}{2}}\right) \sum_{m=1}^{\infty} \frac{mx^m}{1-x^m}. \tag{25.26}$$

He noted that the last sum on the right-hand side of (25.26) could be written as

$$\sum_{m=1}^{\infty} mx^m(1 + x^m + x^{2m} + \cdots) = \sum_{m=1}^{\infty} m \sum_{k=1}^{\infty} x^{mk}. \tag{25.27}$$

He next observed that the coefficient of x^n would contain an m for each $mk = n$. This meant that when the order of summation in (25.27) was changed, the coefficient of x^n would be given by$\sum_{m|n} m = \sigma(n)$. Therefore

$$\sum_{m=1}^{\infty} \frac{mx^n}{1-x^m} = \sum_{n=1}^{\infty} \sigma(n)x^n,$$

and (25.26) could be rewritten as

$$\left(\sum_{n=0}^{\infty} (-1)^n x^{\frac{n(3n\mp1)}{2}}\right) \sum_{n=1}^{\infty} \sigma(n)x^n - \sum_{m=1}^{\infty} (-1)^m \frac{n(3n\mp1)}{2} x^{\frac{m(3n\mp1)}{2}} = 0. \tag{25.28}$$

Euler then multiplied the two sums in (25.28) and equated coefficients to obtain

$$\sigma(n) - \sigma(n-1) - \sigma(n-2) + \sigma(n-5) + \sigma(n-7) - \cdots = 0, \tag{25.29}$$

where the last nonzero term on the right-hand side of (25.29) wold be $\pm n$ if n were a pentagonal number. This proved the formula, on the assumption that (25.25) was true.