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Stochastic Stability of Differential Equations in Abstract Spaces

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Contents

Preface

The aim of this book is to give a basic and systematic account of stability theory of stochastic differential equations in Hilbert spaces and its application to practical stochastic systems such as a stochastic partial differential equation (SPDE).

I have tried to organize the main content into an easily accessible monograph after a brief review of some preliminary material. I begin my account in Chapter 1 by recalling some notions and notations from the theory of stochastic differential equations in Hilbert spaces. Some fundamental concepts such as Q-Wiener process, stochastic integration, strong or mild solution will be reviewed carefully. Most theorems or propositions are stated without proofs here. However, I will present those proofs of results that are not available in the existing books and are to be found scattered in the literature.

Chapter 2 of this book is devoted to a detailed development of stability theory for linear stochastic evolution equations. The central part is a formulation of the characteristic conditions for mean square and almost sure exponential stability in terms of the Lyapunov type of equations.

In Chapter 3, I mainly focus on the development of a stability property for a wide class of nonlinear stochastic differential equations. This chapter contains basic theory and illustrative examples in connection with the stability behavior of nonlinear stochastic systems. In particular, I generalize those linear characteristic results from Chapter 2 in an appropriate way to obtain some nonlinear versions for semi-linear stochastic evolution equations. Motivated by the idea of reducing the stability problem of nonlinear stochastic systems to the corresponding one of linear systems, I develop the so-called Lyapunov function characterization method and explore the associated first-order approximation techniques. Other interesting topics such as non-exponential decay or stabilization by noise of systems are also considered.

Chapter 4 is a statement of stability theory for stochastic functional differential equations. Generally speaking, it is a natural idea to extend the so-called Lyapunov function theory in the previous chapters to develop a Lyapunov functional type of scheme for time delay stochastic systems although this program can frequently become frustrating. To avoid the underlying difficulty, I emphasize in linear cases the so-called fundamental solutions and associated lift-up methods. For nonlinear systems, I introduce various approaches such as a fixed-point theorem or Razumikhin function method to handle stochastic stability problems.

In Chapter 5, I present selected applications in which the choice of the material reflects my own personal preference. Here the treatment is somewhat sketchy and by no means the only way or even the most appropriate way. The purpose of my account is a desire to present some practical topics such as stochastic optimal control or stochastic population dynamics, beyond the main scheme of the book, but having a relation to stability theory of stochastic evolution equations. It is also hoped that the selected presentation here will stimulate further work in these and related fields.

Notes and Comments at the end of each chapter contain historical and related background material as well as references to the results discussed in that chapter. The pervading influence of a variety of authors' work in this book is obvious. I have drawn freely on their work and hopefully I can acknowledge my scientific debts to them by some remarks shown there. It should be emphasized that my choice of material in this book is highly subjective. In particular, this book is organized only to cover stability theory of stochastic models with white noise and a number of specific topics are not treated. For instance, I do not consider stochastic equations driven by a jump process or fractional Brownian motion. For time delay systems, I do not treat stochastic functional differential equations of infinite time lag or neutral type. Each of those subjects would require several additional chapters. On the other hand, the lengthy list of references at the end of the book is somewhat incomplete and only includes those titles which pertain directly to the content. The author wishes to apologize to those researchers whose work might have been overlooked.

The current volume is an outgrowth of the author's book *Stability of Infinite Dimensional Stochastic Differential Equations with Applications* published by Chapman & Hall/CRC in 2006. Since its publication, many new results have appeared and significant progress has been made in this exciting research field. Most ingredients in the author's old version are maintained as the skeleton of the present volume. But they are reorganized and integrated with much material developed in the last decade to reflect new developments and update the existing results.

I would like to thank all those who helped in the realization of this book through encouragement, advice, or scientific exchanges: Jianhai Bao, Joris Bierkens, Tomás Caraballo, María Garrido-Atienza, Anna Kwiecińska, Vidyadhar Mandrekar, Xuerong Mao, Bohdan Maslowski, Michael Scheutzow, Takeshi Taniguchi, Aubrey Truman, Feng-Yu Wang, Wei Wang, George Yin, and Chenggui Yuan. I am extremely grateful to Professor Peter Giblin for his language comments which have led to a significant improvement of my presentation. My special thanks go to Professor Pao-Liu Chow and Professor Jerzy Zabczyk whose warm assistance and concern has provided lasting support at various stages of my academic career. Last but certainly not least, I want to thank my wife and daughter, Lihong and Annie, who had to put up with me being even more “random” during the preparation of this book than usual.