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Preliminaries

We begin by recalling some basic definitions and results, especially those from functional analysis, partial differential equations (PDEs), probability, and stability theories. We review basic notions and notations from deterministic systems and recall important results from stochastic differential equations. We introduce two notions of solutions, mild and strong, for infinite-dimensional stochastic differential equations and consider the existence and uniqueness of solutions under suitable assumptions. We introduce and clarify various definitions of stochastic stability in Hilbert spaces, which are a natural generalization of deterministic stability concepts. To present the proofs of all the results here would require preparatory background material that would significantly increase both the size and scope of this book. Therefore, we adopt the approach of omitting those proofs, which are treated in detail in well-known standard textbooks such as Da Prato and Zabczyk [53], Pazy [187], and Yosida [224]. However, those proofs will be presented that are not available in the existing books and are to be found scattered in the literature, or that discuss ideas specially relevant to our purpose.

1.1 Linear Operators, Semigroups, and Examples

Throughout this book, the sets of nonnegative integers, positive integers, real numbers, and complex numbers are denoted by $\mathbb{N}$, $\mathbb{N}_+$, $\mathbb{R}$, and $\mathbb{C}$, respectively. Also, $\mathbb{R}_+$ denotes the set of all nonnegative real numbers and $\mathbb{R}^n$ denotes the n-dimensional real vector space equipped with the usual Euclidean norm $\|\cdot\|_{\mathbb{R}^n}$, $n \geq 1$. For any $\lambda \in \mathbb{C}$, the symbols $Re\,\lambda$ and $Im\,\lambda$ denote the real and imaginary parts of λ, respectively. Given a set E, the symbol $\mathbf{1}_E$ denotes the characteristic function of E, i.e., $\mathbf{1}_E(x) = 1$ if $x \in E$ and $\mathbf{1}_E(x) = 0$ if $x \notin E$.

A *Banach space* $(X, \|\cdot\|_X)$, real or complex, is a complete normed linear space over $\mathbb{R}$ or $\mathbb{C}$. If the norm $\|\cdot\|_X$ is induced by an inner product $\langle\cdot,\cdot\rangle_X$, then X is called a *Hilbert space*. In this book, we always take the inner product $\langle\cdot,\cdot\rangle_X$ of X to be linear in the first entry and conjugate-linear in the second. We say that a sequence $\{x_n\}_{n\geq 1} \subseteq X$ *(strongly) converges* to $x \in X$ if $\lim_{n\to\infty} \|x_n - x\|_X = 0$. If X contains n linearly independent vectors, but every system of $n+1$ vectors in X is linearly dependent, then X is called an *n-dimensional space*, denoted by $\dim X = n$. Otherwise, the space X is said to be infinite dimensional. We say that X is *separable* if there exists a countable set $S \subseteq X$ such that $\overline{S} = X$, where $\overline{S}$ is the closure of S in X. For a Hilbert space X, a collection $\{e_i\}_{i\geq 1}$ of elements in X is called an *orthonormal set* if $\langle e_i, e_i\rangle_X = 1$ for all i, and $\langle e_i, e_j\rangle_X = 0$ if $i \neq j$. If S is an orthonormal set and no other orthonormal set contains S as a proper subset, then S is called an *orthonormal basis* for X. A Hilbert space is separable if and only if it has a countable orthonormal basis $\{e_i\}$, $i = 1, 2, \ldots$

A typical example of Banach spaces is the so-called Sobolev space, which plays an important role in PDE theory. Let $\mathcal{O}$ be a nonempty domain of $\mathbb{R}^n$, and m be a positive integer. For $1 \leq p < \infty$ we denote by $W^{m,p}(\mathcal{O}; X)$ the set of all elements $y \in L^p(\mathcal{O}; X)$ such that y and its distributional derivatives $\partial^\alpha y$ of order $|\alpha| \leq m$ are in $L^p(\mathcal{O}; X)$, where

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1{}^{\alpha_1} \cdots \partial x_n{}^{\alpha_n}} \qquad \text{and} \qquad |\alpha| = \sum_{i=1}^n \alpha_i.$$

Then $W^{m,p}(\mathcal{O}; X)$ is a Banach space under the norm

$$\|y\|_{m,p} = \left(\int_{\mathcal{O}} \sum_{|\alpha|\leq m} \|\partial^\alpha y(x)\|_X^p dx \right)^{1/p}, \qquad y \in W^{m,p}(\mathcal{O}; X).$$

On the other hand, we denote by $C^m(\mathcal{O}; X)$ the set of all m-times continuously differentiable vectors in $\mathcal{O}$, and by $C_0^m(\mathcal{O}; X)$ the subspace of $C^m(\mathcal{O}; X)$ consisting of those vectors that have compact supports in $\mathcal{O}$. Another important Banach space $W_0^{m,p}(\mathcal{O}; X)$ is defined as the completion of $C_0^\infty(\mathcal{O}; X)$ in the metric of $W^{m,p}(\mathcal{O}; X)$.

In general, the spaces $W^{m,p}(\mathcal{O}; X)$ and $W_0^{m,p}(\mathcal{O}; X)$ do not coincide for bounded $\mathcal{O}$. However, it is true that

$$W^{m,p}(\mathbb{R}^n, \mathbb{R}) = W_0^{m,p}(\mathbb{R}^n, \mathbb{R}).$$

The case $p=2$ is special since the spaces $W^{m,2}(\mathcal{O}; X)$, $W_0^{m,2}(\mathcal{O}; X)$ (frequently written as $H^m(\mathcal{O}; X)$, $H_0^m(\mathcal{O}; X)$) are Hilbert spaces if X is a Hilbert space under the scalar product

$$\langle y, z\rangle_{m,2} = \int_\Omega \sum_{|\alpha|\le m} \langle \partial^\alpha y(x), \partial^\alpha z(x)\rangle_X dx.$$

Let X and Y be two Banach spaces and $\mathscr{D}(A)$ a subspace of X. A map $A\colon \mathscr{D}(A) \subseteq X \to Y$ is called a *linear operator* if the following relation holds:

$$A(\alpha x + \beta y) = \alpha Ax + \beta Ay \qquad \text{for any} \quad x,\ y \in \mathscr{D}(A),\ \ \alpha, \beta \in \mathbb{R} \text{ or } \mathbb{C}.$$

The subspace $\mathscr{D}(A)$ is called the *domain* of A. If A maps any bounded subsets of $\mathscr{D}(A)$ into bounded subsets of Y, we say that A is a *bounded linear operator*. We denote by $\mathscr{L}(X, Y)$ the set of all bounded linear operators A from X to Y with $\mathscr{D}(A) = X$. It may be shown that $\mathscr{L}(X, Y)$ is a Banach space under the operator norm $\|\cdot\|_{\mathscr{L}(X,Y)}$, or simply $\|\cdot\|$, given by

$$\|A\| := \sup_{\|x\|_X\le 1} \|Ax\|_Y = \sup_{\|x\|_X=1} \|Ax\|_Y \qquad \text{for any} \qquad A \in \mathscr{L}(X, Y).$$

For simplicity, we frequently write $\mathscr{L}(X)$ for $\mathscr{L}(X, X)$.

For any linear operator $A\colon \mathscr{D}(A) \subseteq X \to Y$, we define $\mathscr{K}(A) = \{x \in \mathscr{D}(A)\colon Ax = 0\}$ and $\mathscr{R}(A) = \{Ax\colon x \in \mathscr{D}(A)\}$. They are called the *kernel* and *range* spaces of A, respectively.

Theorem 1.1.1 *Let X and Y be two Banach spaces. Then the following results hold:*

(i) (Open Mapping Theorem) $A \in \mathscr{L}(X, Y)$ *and* $\mathscr{R}(A) = Y$ *imply that for any open set* $E \subseteq X$, *the set* $A(E)$ *is open in* Y.
(ii) (Inverse Mapping Theorem) $A \in \mathscr{L}(X, Y)$ *with* $\mathscr{R}(A) = Y$ *and* $\mathscr{K}(A) = \{0\}$ *imply that the inverse operator* A^{-1} *exists and* $A^{-1} \in \mathscr{L}(Y, X)$.
(iii) (Principle of Uniform Boundedness) $\Sigma \subseteq \mathscr{L}(X, Y)$ *and* $\sup_{A\in\Sigma} \|Ax\|_Y < \infty$ *for each* $x \in X$ *imply that* $\sup_{A\in\Sigma} \|A\| < \infty$.

Let $Y = K$ where $K = \mathbb{R}$ or $\mathbb{C}$. Any $f \in \mathscr{L}(X, K)$ is called a *bounded linear functional* on X. In the sequel, we put $X^* = \mathscr{L}(X, K)$, which is a Banach space under the norm $\|\cdot\|_{X^*}$ and call X^* the *dual space* of X. Quite often, we write $f(x)$ for any $f \in X^*$, $x \in X$ by $\langle\langle x, f\rangle\rangle_{X,X^*}$, and the symbol $\langle\langle\cdot,\cdot\rangle\rangle_{X,X^*}$ is referred to as the *duality pair* between X and X^*. The following theorem assures the existence of nontrivial bounded linear functionals on any Banach space.

Theorem 1.1.2 (Hahn–Banach Theorem) *Let X be a Banach space and X_0 a subspace of X. Let $f_0 \in X_0^*$, then there exists an extension $f \in X^*$ of f_0 such that $\|f\|_{X^*} = \|f_0\|_{X_0^*}$.*

Since X^* is a Banach space, we may also talk about the dual space of X^*, i.e., $X^{**} := (X^*)^*$. It is known that for any $x \in X$, by defining

$$x^{**}(f) = f(x) = \langle\langle x, f\rangle\rangle_{X,X^*} \qquad \text{for any} \qquad f \in X^*, \tag{1.1.1}$$

we have $x^{**} \in X^{**}$ and $\|x\|_X = \|x^{**}\|_{X^{**}}$. Thus, the map $x \to x^{**}$ from X into X^{**} is linear and injective and preserves the norm so that X is embeddable into X^{**}. If we regard x exactly the same as x^{**}, then $X \subset X^{**}$. In general, the strict inclusion may hold, a fact that naturally leads to the following definition.

Definition 1.1.3 A Banach space X is said to be *reflexive* if $X = X^{**}$. Precisely, for any $x^{**} \in X^{**}$, there exists an $x \in X$ such that (1.1.1) holds.

The most important class of reflexive spaces are Hilbert spaces, a fact that is justified by the following theorem.

Theorem 1.1.4 (Riesz Representation Theorem) *Let X be a Hilbert space, then $X^* = X$. More precisely, for any $f \in X^*$, there exists a unique element $y \in X$ such that*

$$f(x) = \langle x, y\rangle_X \qquad \textit{for any} \qquad x \in X, \tag{1.1.2}$$

and conversely, for any $y \in X$, by defining f as in (1.1.2), one has $f \in X^*$. It clearly makes sense to write $\langle\cdot,\cdot\rangle_X$ for $\langle\langle\cdot,\cdot\rangle\rangle_{X,X^*}$ on this occasion.

Closed linear operators, generally unbounded, frequently appear in applications, notably in connection with partial differential equations.

Definition 1.1.5 Let X and Y be two Banach spaces. A linear operator $A\colon \mathscr{D}(A) \subseteq X \to Y$ is said to be *closed* if whenever

$$x_n \in \mathscr{D}(A), \ \ n \geq 1, \ \text{ and } \ \lim_{n\to\infty} x_n = x, \ \ \lim_{n\to\infty} Ax_n = y,$$

it follows that $x \in \mathscr{D}(A)$ and $Ax = y$.

For a closed linear operator $A\colon \mathscr{D}(A) \subseteq X \to X$, it can be shown that the domain $\mathscr{D}(A)$ is a Banach space under the graph norm $\|x\|_{\mathscr{D}(A)} := \|x\|_X + \|Ax\|_X$, $x \in \mathscr{D}(A)$. It is easy to see that any bounded linear operator having a closed domain is closed. The converse statement can be true in the following sense.

Theorem 1.1.6 (Closed Graph Theorem) *Suppose that $A\colon \mathscr{D}(A) \subseteq X \to Y$ is a closed linear operator. If $\mathscr{D}(A)$ is closed in X, then operator A is bounded.*

In general, it is difficult to prove that an operator is closed. The next theorem states that if this operator is the algebraic inverse of a bounded linear operator, then it is closed.

Theorem 1.1.7 *Assume that X and Y are Banach spaces and let A be a linear operator from X to Y. If A is invertible with $A^{-1} \in \mathscr{L}(Y, X)$, then A is a closed linear operator.*

Let X and Y be two Banach spaces and a linear operator $A: \mathscr{D}(A) \subseteq X \to Y$ is called *densely defined* if $\overline{\mathscr{D}(A)} = X$. If A is densely defined, we may define *Banach space adjoint operator* $A': \mathscr{D}(A') \subseteq Y^* \to X^*$ of A in the following manner. Let

$$\mathscr{D}(A') = \left\{ y^* \in Y^*: \, y^* A \text{ is continuous on } \mathscr{D}(A) \right\}.$$

The linear operator $A': \mathscr{D}(A') \subseteq Y^* \to X^*$ is defined by

$$\langle\!\langle x, A'y^* \rangle\!\rangle_{X, X^*} = \langle\!\langle Ax, y^* \rangle\!\rangle_{Y, Y^*} \qquad \text{for any} \qquad y^* \in \mathscr{D}(A'), \;\; x \in \mathscr{D}(A).$$

It turns out that A' is uniquely defined and closed and map $A \to A'$ is linear.

Now let us consider the case where A is a densely defined linear operator on a Hilbert space X. Then the Banach space adjoint A' of A is a mapping from X^* into itself. Let $\iota: X \to X^*$ be the map that assigns, for each $x \in X$, the bounded linear functional $\langle \cdot, x \rangle_X$ in X^*. Then ι is a linear isometry, which is surjective by the Riesz Representation Theorem. Now define a map $A^*: X \to X$ by

$$A^* = \iota^{-1} A' \iota.$$

Then $A^*: X \to X$ satisfies

$$\langle Ay, x \rangle_X = (\iota x)(Ay) = (A'\iota x)(y) = \langle y, \iota^{-1} A' \iota x \rangle_X = \langle y, A^* x \rangle_X$$

for any $y \in \mathscr{D}(A)$, $x \in \mathscr{D}(A^*)$, and A^* is called the *Hilbert space adjoint*, or simply *adjoint*, of A. In general, $A^* \neq A'$. However, if X is a real Hilbert space, then $A^* = A'$.

Definition 1.1.8 Let X be a Hilbert space. A densely defined linear operator $A: \mathscr{D}(A) \subseteq X \to X$ is *symmetric* if for all $x, y \in \mathscr{D}(A)$, $\langle Ax, y \rangle_X = \langle x, Ay \rangle_X$. A symmetric operator A is called *self-adjoint* if $\mathscr{D}(A^*) = \mathscr{D}(A)$.

All bounded and symmetric operators are self-adjoint. It may be shown that the adjoint of a densely defined linear operator on a Hilbert space X is closed, and so is every self-adjoint operator. A linear operator A on the Hilbert space X is called *nonnegative*, denoted by $A \geq 0$, if $\langle Ax, x \rangle_X \geq 0$ for all $x \in \mathscr{D}(A)$. It is called *positive* if $\langle Ax, x \rangle_X > 0$ for all non zero $x \in \mathscr{D}(A)$ and *coercive* if

$\langle Ax, x\rangle_X \geq c\|x\|_X^2$ for some $c > 0$ and all $x \in \mathscr{D}(A)$. We denote the spaces of all nonnegative, positive, and coercive operators by $\mathscr{L}^+(X)$, $\mathscr{L}_0^+(X)$, and $\mathscr{L}_c^+(X)$, respectively. A linear operator B is called the *square root* of A if $B^2 = A$.

Theorem 1.1.9 *Let A be a linear operator on the Hilbert space X. If A is self-adjoint and nonnegative, then it has a unique square root, denoted by $A^{1/2}$, which is self-adjoint and nonnegative such that $\mathscr{D}(A) \subset \mathscr{D}(A^{1/2})$. Furthermore, if A is positive, so is $A^{1/2}$.*

Theorem 1.1.10 *Suppose that A is self-adjoint and nonnegative on the Hilbert space X. Then A is coercive if and only if it has a bounded inverse $A^{-1} \in \mathscr{L}(X)$. In this case, A^{-1} is self-adjoint and nonnegative.*

In the family of all bounded linear operators, there is a subclass, called compact operators, which are in many ways analogous to linear operators in finite-dimensional spaces.

Definition 1.1.11 Let X and Y be two Banach spaces. An operator $A \in \mathscr{L}(X, Y)$ is *compact* if for any bounded sequence $\{x_n\}_{n\geq 1}$ in X, the sequence $\{Ax_n\}_{n\geq 1}$ has a convergent subsequence in Y.

Let X be a separable Hilbert space and $\{e_i\}_{i=1}^\infty$ an orthonormal basis. Then for any nonnegative operator $A \in \mathscr{L}(X)$, we define $Tr(A) = \sum_{i=1}^\infty \langle e_i, Ae_i\rangle_X$. The number $Tr(A)$ is called the *trace* of A and is independent of the orthonormal basis chosen. An operator $A \in \mathscr{L}(X)$ is called *trace class* if $Tr(|A|) < \infty$, where $|A| = (A^*A)^{1/2}$. If we endow the trace norm $\|A\|_1 := Tr(|A|)$ for any trace class operator A, then the associated family $\mathscr{L}_1(X)$ of all trace class operators forms a Banach space. An operator $A \in \mathscr{L}(X)$ is called *Hilbert–Schmidt* if $Tr(A^*A) < \infty$. The norm corresponding to a Hilbert–Schmidt inner product is $\|A\|_2 := (Tr(A^*A))^{1/2}$ under which all the Hilbert–Schmidt operators form a Hilbert space $\mathscr{L}_2(X)$. It is easy to show that the following inclusions hold and they are all proper when X is infinite dimensional:

$$\{\text{trace class}\} \subset \{\text{Hilbert–Schmidt}\} \subset \{\text{compact}\}.$$

An operator $A \in \mathscr{L}(X)$ is said to have *finite trace* if the series

$$\sum_{i=1}^\infty \langle e_i, Ae_i\rangle_X < \infty \tag{1.1.3}$$

for any orthonormal basis $\{e_i\}_{i\geq 1}$ in X. In general, it is not true that

$$\sum_{i=1}^{\infty} |\langle e_i, Ae_i\rangle_X| < \infty \tag{1.1.4}$$

for some orthonormal basis implies that $A \in \mathscr{L}_1(X)$. However, for a trace class operator A the sum in (1.1.3) is absolutely convergent and independent of the choice of the orthonormal basis. In particular, for a nonnegative operator $A \in \mathscr{L}(X)$, the concept of a trace class operator coincides with that of an operator having finite trace.

Let $A\colon \mathscr{D}(A) \subseteq X \to X$ be a linear operator on a Banach space X. The *resolvent set* $\rho(A)$ of A is the set of all complex numbers $\lambda \in \mathbb{C}$ such that $(\lambda I - A)^{-1}$ exists and $(\lambda I - A)^{-1} \in \mathscr{L}(X)$, where I is the identity operator on X. For $\lambda \in \rho(A)$, we write $R(\lambda, A) = (\lambda I - A)^{-1}$ and call it the *resolvent operator* of A. The *spectrum* of A is defined to be $\sigma(A) = \mathbb{C} \setminus \rho(A)$. It may be shown that the resolvent set $\rho(A)$ is open in $\mathbb{C}$.

Definition 1.1.12 Let A be a linear operator on Banach space X. Define

(i) $\sigma_p(A) = \{\lambda \in \mathbb{C}\colon \lambda I - A$ is not injective$\}$, and $\sigma_p(A)$ is called the *point spectrum* of A. Moreover, each $\lambda \in \sigma_p(A)$ is called the *eigenvalue*, and each nonzero $x \in \mathscr{D}(A)$ satisfying $(\lambda I - A)x = 0$ is called the *eigenvector* of A corresponding to λ.
(ii) $\sigma_c(A) = \{\lambda \in \mathbb{C}\colon \lambda I - A$ is injective, $\mathscr{R}(\lambda I - A) \neq X$ and $\overline{\mathscr{R}(\lambda I - A)} = X\}$, and $\sigma_c(A)$ is called the *continuous spectrum* of A.
(iii) $\sigma_r(A) = \{\lambda \in \mathbb{C}\colon \lambda I - A$ is injective and $\overline{\mathscr{R}(\lambda I - A)} \neq X\}$, and $\sigma_r(A)$ is called the *residual spectrum* of A.

From this definition, it is immediate that $\sigma_p(A)$, $\sigma_c(A)$, and $\sigma_r(A)$ are mutually exclusive and their union is $\sigma(A)$. If A is self-adjoint, we have $\sigma_r(A) = \emptyset$. Note that if $\dim X < \infty$, all the linear operators A on X are compact and in this case $\sigma(A) = \sigma_p(A)$, a fact that is extendable to any compact operators in infinite-dimensional spaces.

Theorem 1.1.13 *Let X be a Banach space with $\dim X = \infty$. If $A \in \mathscr{L}(X)$ is compact, then one and only one of the following cases holds:*

(i) $\sigma(A) = \{0\}$;
(ii) $\sigma(A) = \{0, \lambda_1, \ldots, \lambda_n\}$ *where for each* $1 \leq k \leq n$, $\lambda_k \neq 0$ *and* λ_k *is an eigenvalue of* A;
(iii) $\sigma(A) = \{0, \lambda_1, \lambda_2, \ldots\}$ *where for each* $k \geq 1$, $\lambda_k \neq 0$ *and* λ_k *is an eigenvalue of* A *with* $\lim_{k\to\infty} \lambda_k = 0$.

In this book, we shall employ the theory of linear semigroups, which usually allows a uniform treatment of many systems such as some parabolic, hyperbolic, and delay equations.

Definition 1.1.14 A *strongly continuous or C_0-semigroup* $S(t) \in \mathscr{L}(X)$, $t \geq 0$, on a Banach space X is a family of bounded linear operators $S(t)\colon X \to X$, $t \geq 0$, satisfying the following:

(i) $S(0)x = x$ for all $x \in X$;
(ii) $S(t+s) = S(t)S(s)$ for all $t, s \geq 0$;
(iii) $S(t)$ is strongly continuous, i.e., for any $x \in X$, $S(\cdot)x\colon [0,\infty) \to X$ is continuous.

For any C_0-semigroup $S(t)$ on X, there exist constants $M \geq 1$ and $\mu \in \mathbb{R}$ such that

$$\|S(t)\| \leq Me^{\mu t}, \qquad t \geq 0. \tag{1.1.5}$$

In particular, the semigroup $S(t)$ is called (*uniformly*) *bounded* if $\mu = 0$ and *exponentially stable* if $\mu < 0$. The semigroup $S(t)$, $t \geq 0$, is called *eventually norm continuous* if the map $t \to S(t)$ is continuous from (r,∞) to $\mathscr{L}(X)$ for some $r > 0$. In particular, $S(t)$, $t \geq 0$, is simply called *norm continuous* if the map $t \to S(t)$ is continuous from $(0,\infty)$ to $\mathscr{L}(X)$. If $M = 1$ in (1.1.5), the semigroup $S(t)$, $t \geq 0$, is called a *pseudo contraction* C_0-semigroup, and if further $\mu = 0$, it is called a *contraction* C_0-semigroup.

In association with the C_0-semigroup $S(t)$, we may define a linear operator $A\colon \mathscr{D}(A) \subseteq X \to X$ by

$$\mathscr{D}(A) = \left\{x \in X\colon \lim_{t\downarrow 0} \frac{S(t)x - x}{t} \text{ exists in } X\right\},$$

$$Ax = \lim_{t\downarrow 0} \frac{S(t)x - x}{t}, \quad x \in \mathscr{D}(A).$$

The operator A is called the *infinitesimal generator*, or simply *generator*, of the semigroup $\{S(t)\}_{t\geq 0}$, which is frequently written as e^{tA}, $t \geq 0$, in this book. It may be shown that A is densely defined and closed.

For an arbitrary C_0-semigroup e^{tA}, $t \geq 0$, the following theorem gives a characterization of its generator A.

Theorem 1.1.15 (Hille–Yosida Theorem) *Let X be a Banach space and $A\colon \mathscr{D}(A) \subseteq X \to X$ be a linear operator. Then the following are equivalent:*

(i) A generates a C_0-semigroup e^{tA}, $t \geq 0$, on X such that (1.1.5) holds for some $M \geq 1$ and $\mu \in \mathbb{R}$.

(ii) A is densely defined, closed, and there exist constants $\mu \in \mathbb{R}$, $M \geq 1$ such that $\rho(A) \supset \{\lambda \in \mathbb{C} : Re\,\lambda > \mu\}$ and

$$\|R(\lambda, A)^n\| \leq \frac{M}{(Re\,\lambda - \mu)^n} \quad \textit{for any} \quad n \in \mathbb{N}_+, \ Re\,\lambda > \mu. \tag{1.1.6}$$

In general, it is not easy to verify (1.1.6) for each $n \in \mathbb{N}_+$. We can give, however, a simple characterization of linear operators that generate pseudo contraction C_0-semigroups.

Definition 1.1.16 A linear operator $A : \mathscr{D}(A) \subset X \to X$ on a Banach space X is called *dissipative* if

$$\|(\lambda I - A)x\|_X \geq \lambda \|x\|_X \quad \text{for all} \quad x \in \mathscr{D}(A) \quad \text{and} \quad \lambda > 0.$$

Theorem 1.1.17 (Lumer and Phillips Theorem) *Let $A : \mathscr{D}(A) \subset X \to X$ be a linear operator defined on X. Then A is the generator of a contraction C_0-semigroup on X if and only if*

(i) A is a closed linear operator with dense domain in X;
(ii) A and its adjoint operator A' are dissipative.

If X is a Hilbert space, the conditions in Theorem 1.1.17 may be simplified. In particular, we have the following proposition, which is a consequence of Theorem 1.1.17.

Proposition 1.1.18 *Let A be a closed, densely defined linear operator on a Hilbert space X. There exists a real number $\alpha \in \mathbb{R}$ such that*

$$Re\,\langle x, Ax\rangle_X \leq \alpha \|x\|_X^2 \quad \textit{for all} \quad x \in \mathscr{D}(A), \tag{1.1.7}$$

and

$$Re\,\langle x, A^*x\rangle_X \leq \alpha \|x\|_X^2 \quad \textit{for all} \quad x \in \mathscr{D}(A^*), \tag{1.1.8}$$

if and only if A generates a pseudo contraction C_0-semigroup e^{tA}, $t \geq 0$, satisfying

$$\|e^{tA}\| \leq e^{\alpha t} \quad \textit{for all} \quad t \geq 0. \tag{1.1.9}$$

We state some properties of C_0-semigroups and their generators.

Proposition 1.1.19 *Let e^{tA}, $t \geq 0$, be a C_0-semigroup on a Banach space X and $A_n = nAR(n, A) \in \mathscr{L}(X)$, $n \in \rho(A)$, called the Yosida approximation of A. Then*

$$\lim_{n\to\infty} \|A_n x - Ax\|_X = 0 \quad \textit{for any} \quad x \in \mathscr{D}(A),$$

and

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|e^{tA_n}x - e^{tA}x\|_X = 0 \quad \textit{for any} \quad x \in X, \quad T \geq 0.$$

Proposition 1.1.20 *For the generator A of a C_0-semigroup e^{tA}, $t \geq 0$, on a Banach space X,*

(i) if $x \in \mathscr{D}(A)$, then $e^{tA}x \in \mathscr{D}(A)$ and

$$\frac{d}{dt}e^{tA}x = e^{tA}Ax = Ae^{tA}x \quad \textit{for all} \quad t \geq 0;$$

(ii) for every $t \geq 0$ and $x \in X$,

$$\int_0^t e^{sA}x\,ds \in \mathscr{D}(A) \quad \textit{and} \quad A\int_0^t e^{sA}x\,ds = e^{tA}x - x.$$

Let X be a Banach space and consider the following deterministic linear Cauchy problem on X,

$$\begin{cases} \dfrac{dy(t)}{dt} = Ay(t), & t \geq 0, \\ y(0) = y_0 \in X, \end{cases} \tag{1.1.10}$$

where A is a linear operator that generates a C_0-semigroup e^{tA}, $t \geq 0$, on X. If $y_0 \in \mathscr{D}(A)$, we have by Proposition 1.1.20 that $e^{tA}y_0 \in \mathscr{D}(A)$ and

$$\frac{d}{dt}(e^{tA}y_0) = Ae^{tA}y_0, \qquad t \geq 0. \tag{1.1.11}$$

Hence, $y(t) = e^{tA}y_0$, $t \geq 0$, is a solution of the differential equation (1.1.10). If $y_0 \notin \mathscr{D}(A)$, the equality (1.1.11) may not be meaningful. However, for any $y_0 \in X$ it does make sense to define $y(t) = e^{tA}y_0$, $t \geq 0$, which is called a *mild solution* of (1.1.10). Quite a few partial differential equations can be formulated in the form (1.1.10).

Example 1.1.21 Let $\{\lambda_i\}$ be a sequence of complex numbers and $\{e_i\}$, $i \in \mathbb{N}_+$, be an orthonormal basis in a separable Hilbert space H. We define on H an operator A by

$$Ax = \sum_{i=1}^{\infty} \lambda_i \langle x, e_i\rangle_H e_i, \qquad x \in \mathscr{D}(A),$$

with its domain

$$\mathscr{D}(A) = \left\{ x \in H \colon \sum_{i=1}^{\infty} |\lambda_i \langle x, e_i\rangle_H|^2 < \infty \right\}.$$