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INTRODUCTION TO VOLUME II

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The present volume is the second in a series of collections of mainly expository articles on the arithmetic theory of automorphic forms. The books are primarily intended for two groups of readers. The first group is interested in the structure of automorphic forms on reductive groups over number fields, and specifically in qualitative information about the multiplicities of automorphic representations. The second group is interested in the problem of classifying l-adic representations of Galois groups of number fields. Langlands' conjectures elaborate on the notion that these two problems overlap to a considerable degree. The goal of this series of books is to gather into one place much of the evidence that this is the case, and to present it clearly and succinctly enough so that both groups of readers are not only convinced by the evidence but can pass with minimal effort between the two points of view.

The first volume mainly dealt with the mechanics of the stable trace formula, with special emphasis on the role of the Fundamental Lemma, whose proof had recently appeared. The present volume is largely concerned with application of the methods of arithmetic geometry to the theory of Shimura varieties. The primary motivation, in both cases, is the construction of the compatible families of ℓ-adic Galois representations attached to cohomological automorphic representations π of $GL(n)$ over totally imaginary quadratic extensions of totally real number fields – CM fields, in other words. These are the Galois representations that allow the most direct generalizations of methods developed for the study of the arithmetic of elliptic curves over $\mathbb{Q}$. This is one reason they have increasingly attracted the attention of algebraic number theorists. The latter belong to the second group of readers to whom this book is addressed.

Most of the Galois representations that can be attached to automorphic representations are realized in the ℓ-adic cohomology of Shimura varieties, with coefficients in local systems. The Shimura varieties to be considered in this book are attached to unitary similitude groups; they belong to the class of *PEL Shimura varieties*, the first large family of locally symmetric varieties studied by Shimura in his long series of papers in the early 1960s. The abbreviation stands for Polarization-Endomorphisms-Level; like the other PEL Shimura varieties, the ones studied here are moduli spaces of polarized abelian varieties of a fixed dimension, together with an algebra of endomorphisms and a fixed level structure, all of which have to satisfy certain compatibilities. If one drops the endomorphisms one is left with the moduli space of polarized abelian varieties of fixed dimension g with level structure, in other words with the familiar Siegel moduli space. This is the Shimura variety attached to the group $GSp(2g)$ of similitudes of a $2g$-dimensional vector space endowed with a non-degenerate

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alternating form. The decision to focus instead on unitary similitude groups was dictated by the close relation between $GL(n)$ and unitary groups, which is responsible for the role of the Shimura varieties considered here in the construction of the Galois representations attached to $GL(n)$ over a totally real or CM field. A detailed treatment of Siegel modular varieties can be found in Morel's two articles [M08, M11].

The Shimura varieties considered in this book are all *proper* as complex algebraic varieties; the corresponding unitary similitude groups are anisotropic modulo center. This choice is sufficient in order to attach Galois representations to the automorphic representations that contribute to the cohomology[1] of the adèlic locally symmetric space attached to $GL(n)$. The Galois representations discussed in chapters 8-11 of this Volume, and in Chapter CHL.IV.C of Volume 1, are directly attached to the *polarized* cohomological cuspidal automorphic representations of $GL(n)$; these are the ones that admit descents to unitary groups. Galois representations can also be attached to cuspidal automorphic representations of $GL(n)$ that are cohomological but not polarized [HLTT, Sch, B]; their construction is based on techniques from p-adic geometry that are beyond the scope of the present volume.

A. Zeta functions of unitary Shimura varieties

Let $\mathcal{K}$ be a totally imaginary quadratic extension of a totally real field F. Let $(V, [\cdot,\cdot])$ be a non-degenerate hermitian space for $\mathcal{K}/F$, i.e., V is an n-dimensional $\mathcal{K}$-vector space and $[\cdot,\cdot]$ is a $\mathbf{c}$-sesquilinear form satisfying $[x, y] = \mathbf{c}([y, x])$ for all $x, y \in V$. Let $GU(V)$ denote the group of similitudes of $(V, [\cdot,\cdot])$, and let $G \subset GU(V)$ be the subgroup of similitudes with rational similitude factor, defined as in the chapters of Genestier-Ngô, Nicole, Rozensztajn, and Zhu. This is viewed as an algebraic group over $\mathbb{Q}$. Let $K_f \subset G(\mathbf{A}_f)$ be an open compact subgroup, and let $K_\infty \subset G(\mathbb{R})$ be a maximal connected subgroup that contains the center of $G(\mathbb{R})$ and is compact modulo the center. Then $X := G(\mathbb{R})/K_\infty$ is the union of finitely many copies of a hermitian symmetric domain, and the double coset space

$$G(\mathbb{Q})\backslash G(\mathbf{A})/K_\infty \times K_f$$

can be identified with the set of points of a complex quasi-projective variety $Sh(G, X)_{K_f}$. In Chapter 1 and (in more detail) in Chapter 2, it is explained that X is endowed with a canonical structure, following Deligne, that does not depend on the choice of K_∞, and the Shimura variety $Sh(G, X)_{K_f}$ has a model over the *reflex field* E, a number field determined by the pair (G, X) and contained in the Galois closure of $\mathcal{K}$ over $\mathbb{Q}$. This model identifies $Sh(G, X)_{K_f}$ with (a part of) the moduli space of quadruples $(A, \iota, \lambda, \bar{\eta})$, where A is an abelian variety of dimension n, λ is a polarization of A, ι is an action of an appropriate order in $\mathcal{K}$ on A, and $\bar{\eta}$ is a level structure depending on K_f. The quadruple $(A, \iota, \lambda, \bar{\eta})$ are moreover required to satisfy certain compatibilities that were introduced by Shimura in his definition of *PEL types* (for Polarization, Endomorphism, Level); we refer the reader to Chapter 2 for details.

We always assume that the derived subgroup $G^{der} \subset G$ is an *anisotropic*, which equivalent to the condition $[x, x] \neq 0$ for all $x \in V \setminus \{0\}$. If $n > 2$, this is equivalent

[1] Here and below, by *cohomology* of a locally symmetric space we will understand the cohomology with coefficients in local systems attached to algebraic representations of the corresponding reductive group.

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to the condition that, for some complex embedding $\tau : \mathcal{K} \hookrightarrow \mathbb{C}$, the induced hermitian form on $V \otimes_{\mathcal{K},\tau} \mathbb{C}$ is positive- or negative-definite. Under this hypothesis, it is a well-known theorem of Borel and Harish-Chandra that the double coset space $G(\mathbb{Q})\backslash G(\mathbf{A})/K_\infty \times K_f$ is a compact complex variety, and thus the Shimura variety $Sh(G, X)_{K_f}$ is projective.

Suppose the group G is unramified at p; in other words, it is quasi-split over $\mathbb{Q}_p$ and splits over an unramified extension of $\mathbb{Q}_p$. Then $G(\mathbb{Q}_p)$ contains a hyperspecial maximal compact subgroup given as the stabilizer of some $\Lambda \subset V(\mathbb{Q}_p)$, a lattice that is self-dual relative to $[\cdot,\cdot]$. We suppose that $K_f = K_p \times K^p$ where K_p is such a hyperspecial maximal compact subgroup of $G(\mathbb{Q}_p)$; moreover, we suppose that K^p is a sufficiently small compact open subgroup of the adeles of $\mathbb{Q}$ away from p. In Chapter 3 it is explained that the Shimura variety $Sh(G, X)_{K_f}$ then has a *smooth* model S_{K^p} over the integers $\mathcal{O}_\mathfrak{p}$ in the completion of E at any p-adic place $\mathfrak{p}$; moreover, under our hypothesis that G^{der} is anisotropic, it is known that S_{K^p} is *projective* over $\mathcal{O}_\mathfrak{p}$.

Remark. (Contributed by Y. Zhu) For the smooth quasi-projective integral model of the PEL Shimura variety considered by Kottwitz (at a hyperspecial prime), it is known that it is projective if and only if G^{der} is anisotropic over $\mathbb{Q}$. A special case of this assertion, when $End_B(V)$ is a division algebra, is proved by Kottwitz on p. 392 of [K92a]. The general case follows from the main results of Lan's thesis [Lan13]. In fact, Lan constructs smooth projective toroidal compactifications of the integral model, whose boundary strata are also smooth, see [Lan13, Theorem 4.1.1.1, Theorem 7.3.3.4]. Thus the projectivity of Kottwitz's integral model is equivalent to the projectivity of the generic fiber. But for the generic fiber, it is well known that projectivity is equivalent to G^{der} being anisotropic (see for instance [Pink]). We mention that the same projectivity criterion is also valid for PEL integral models of more general levels. This can be deduced from the results of [Lan11], cf. the discussion on p. 7 of [Pera]. We note that many of Lan's results have been generalized by Madapusi Pera to Hodge type, in [Pera]. See especially [Pera, Corollary 4.1.7] for the same projectivity criterion for the Hodge-type integral models.

The computation of the local factor of the zeta function at $\mathfrak{p}$ comes down to a parametrization of the fixed points, on the special fiber of S_{K^p}, of a correspondence T_a obtained by composing a Hecke operator T (at a prime not dividing p) with the power $Frob_\mathfrak{p}^a$, $a >> 0$, of the (geometric) Frobenius automorphism at $\mathfrak{p}$. The points of the special fiber are partitioned into *isogeny classes* of PEL quadruples $(A, \iota, \lambda, \bar{\eta})$ – in other words, isogeny classes of abelian varieties A, together with the additional structures – and each such isogeny class is preserved by T_a. Thus the determination of the local factor can be divided into four steps. This division is artificial and does not follow the actual proof, and the account given below is a gross oversimplification; however, it works as a first approximation. Precise statements can be found in Chapter 4, specifically Theorem 5.3.1.

(i) Parametrization of the set of isogeny classes

The crucial observation is that the Honda-Tate classification of abelian varieties over finite fields, in terms of their Frobenius automorphisms, allows us to parametrize the set $\mathcal{V}_\mathcal{K}$ of isogeny classes of PEL quadruples by a certain subset of elliptic conjugacy classes in $I(\mathbb{Q})$, where I runs through a certain family of reductive group over $\mathbb{Q}$ whose conjugacy classes

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can be related to those of G. In this way the isogeny class supplies an elliptic conjugacy class $[\gamma_0] \subset G(\mathbb{Q})$.

(ii) Parametrization of an individual isogeny class

The next step is to parametrize the fixed points belonging to the isogeny class corresponding to the conjugacy class $[\gamma_0]$. Suppose for a moment that A is an abelian variety over a finite field k of characteristic p. Then for any prime $\ell \neq p$, the ℓ-adic Tate module $T_\ell(A)$ is canonically a lattice in $H_1(A_{\bar{k}}, \mathbb{Q}_\ell)$, defined as the dual of the ℓ-adic cohomology $H^1(A_{\bar{k}}, \mathbb{Q}_\ell)$. An A' related to A by a prime-to-p-isogeny then defines by duality a lattice in

$$H^1(A_{\bar{k}}, \mathbf{A}_f^p) := \prod_{\ell \neq p}{}' H^1(A_{\bar{k}}, \mathbb{Q}_\ell),$$

the restricted product being taken with respect to the integral cohomology. In this way the prime-to-p isogeny classes of PEL quadruples can be related to $G(\mathbf{A}_f^p) = \prod'_{\ell \neq p} G(\mathbb{Q}_\ell)$-orbits in the space of lattices in $H^1(A_{\bar{k}}, \mathbf{A}_f^p)$. A standard construction relates fixed points of Hecke correspondences on this set of orbits with orbital integrals in $G(\mathbf{A}_f^p)$.

Incorporating p-power isogenies is more subtle. Instead of ℓ-adic étale cohomology, one needs to classify Frobenius-stable lattices in the rational Dieudonné module of A. Since the Frobenius operator is only σ-linear, where σ is a generator of the Galois group of the maximal unramified extension $\mathbb{Q}_p^{un}$ of $\mathbb{Q}_p$, it is not surprising that this part of the classification leads to *twisted* orbital integrals over finite unramified extensions of $\mathbb{Q}_p$.

In this way, the global conjugacy class $[\gamma_0]$ of Step A is joined by a pair (γ, δ) with $\gamma \in G(\mathbf{A}_f^p)$ and $\delta \in G(\mathbb{Q}_p{}^{un})$. The triple $(\gamma_0; \gamma, \delta)$ is called a *Kottwitz triple* if it satisfies the axioms of Definition 4.1.1 of Chapter 4.

(iii) Reconciliation of the global and local data by Galois cohomology

Steps (i) and (ii) have established a map from the set of points of S_{K^p} over finite fields to the set of Kottwitz triples. Thus the Lefschetz trace of the correspondence T_a is a weighted sum over the set of triples in the image of this map. Perhaps the deepest point in the Langlands-Kottwitz method is the result of Kottwitz that asserts that the image consists precisely of those triples for which a certain Galois cohomological invariant (which is naturally known as the *Kottwitz invariant*) vanishes. A series of reductions translates this fundamental observation into an application of the comparison theorem of p-adic Hodge theory. For all this, see Chapter 1, Proposition 6.3.1, and Chapter 4, §4, especially Theorem 4.2.8 and the sketch of its proof.

The Kottwitz invariant of a triple $(\gamma_0; \gamma, \delta)$ closely resembles the cohomological invariant that is the subject of Theorem 4.5 (also due to Kottwitz) of Chapter [H.I.A] of [CHLN]. The most obvious difference – that the Kottwitz invariant attached to points on Shimura varieties involves a twisted conjugacy class at p – turns out (again thanks to Kottwitz) to be a manageable problem. Thus in the end the Lefschetz trace of T_a can be written, just as in

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formula (4.7) of Chapter [H.I.A] of [CHLN], as a weighted sum of adelic orbital integrals, indexed by global conjugacy classes. A similar analysis applies to compute the Lefschetz trace of the operator T_a acting on the cohomology of an ℓ-adic local system $\mathcal{W}_\rho$ defined by an algebraic representation ρ of G.

The fourth step is the comparison of the formula that results from steps (i)-(iii) with the stabilized trace of a Hecke operator T'_a acting on the space of automorphic forms $L_2(G(\mathbb{Q})\backslash G(\mathbf{A})/Z)$, where $Z \subset G(\mathbb{R})$ is a subgroup of the center chosen to make the quotient space compact (for example, one can take Z to be the subgroup $\mathfrak{A}_G$ introduced in Chapter L.IV.A of Volume 1). The operators T'_a and T_a coincide at finite places prime to p. At p the local component of T'_a is a Hecke operator determined explicitly by a and the Shimura datum, while the archimedean component of T'_a is a discrete series pseudocoefficient determined by ρ, as in Chapter CHL.IV.B of [CHLN]. In this way the characteristic polynomial of $Frob_\mathfrak{p}$ on eigenspaces of the Hecke operators away from p is determined by the traces of Hecke operators at p on these same eigenspaces, confirming the Langlands conjecture in the situations where the method applies.

The sketch given here is loosely based on two fundamental papers of Kottwitz: The idea of comparing the Lefschetz trace formula for points of Shimura varieties over finite fields with the Selberg trace formula for automorphic forms first appeared in a paper by Ihara [I67], where it was applied to modular curves. Ihara's method was extended by Langlands to the adelic setting, allowing for certain kinds of bad reduction, in [L73]. Langlands wrote a series of papers on the subject in the 1970s, and formulated a conjecture on the form of the zeta function of a general Shimura variety in [L79]. Over the next decade Kottwitz developed the techniques of Galois cohomology, in parallel with those introduced in Volume 1 for the purposes of stabilizing the trace formula, in the end obtaining a formula for the Lefschetz traces of the operators T_a with the same shape as the automorphic trace of T'_a. Readers of Volume 1 will recall that the stabilization of the elliptic part of the Arthur-Selberg trace formula breaks down into three steps analogous to (A-C) above. After the first two steps, the (elliptic part of the) geometric side of the trace formula can be written as a sum over conjugacy classes of pairs (γ, γ_v), where γ (resp. γ_v) is a global (resp. adelic) conjugacy class, satisfying some cohomological restrictions. The analogous formula was proved by Kottwitz in [K92a] for PEL type Shimura varieties attached to groups G whose Lie algebras are of type A or C.

The formula for Lefschetz traces was rewritten in [K90] as a sum over stable global conjugacy classes of purely local orbital integrals, at the cost of introducing endoscopic groups, and assuming the Fundamental Lemma and related conjectures. The resulting formula was completely analogous to the elliptic part of the stable trace formula. While everything developed in the first part of the present book is at least implicit in [K90], the latter is not so easy to use as a reference, because his formulation of the answer makes no distinction between conjectures, like the Fundamental Lemma, that have since been proved, and the most general conjectures relative to the parametrization of points on general Shimura varieties. Moreover, Kottwitz went further: he rewrote the spectral side of the trace formula in terms of Arthur parameters, defined in terms of the Langlands group whose existence is still hypothetical. In Arthur's book [A13] on classical groups, as

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(implicitly) in the chapters by Clozel, Harris, and Labesse in Volume 1, cuspidal automorphic representations of $GL(n)$ are used as a substitute for Arthur parameters. One of the aims of the Paris book project was to provide a usable reference for the applications of the trace formula to the Shimura varieties most relevant to the construction of Galois representations. Since these Galois representations are attached to (polarized) cohomological cuspidal automorphic representations of $GL(n)$, it is completely natural to use these representations as parameters.

Remark. There are other good introductions to this subject matter, notably Milne's three papers [Mi90, Mi92, Mi05]. The treatment here differs from that of Milne in that, as in Volume 1, we focus specifically on the case of Shimura varieties attached to unitary similitude groups. This is the most important case for the construction of Galois representations; moreover, in this case, the elliptic conjugacy classes that arise in the trace formula, as well as the transfer factors used to stabilize the trace formula, are easy to describe in terms of algebraic number theory (see §8 of the first chapter of [CHLN]).

Unitary groups vs. similitude groups

The Shimura variety attached to the (rational) similitude group $GU(V)$ is of PEL type. In particular, it is (a piece of) the solution of a moduli problem that has been well understood in characteristic zero since Shimura's work of the 1960s, and the *determinant condition* introduced by Kottwitz in [K92a] (see Chapter 3, section 2.2) provides a definition of the moduli problem that is valid in all good characteristics. This makes it possible to treat the parametrization of the points efficiently, but the reduction of the automorphic theory of $GU(V)$ to that of $U(V)$, and thus (by base change) to that of $GL(n)$, is not trivial; see §1 of Chapter CHL.IV.C of [CHLN] for an illustration of this. Even when this issue has been resolved, the parametrization of automorphic representations of $GU(V)$ requires an auxiliary Hecke character that is destined to disappear when constructing automorphic Galois representations, but that in the intermediate steps introduces additional variables that pose an ultimately useless challenge to fitting the crucial formulas in a single line.

It is also possible to attach a Shimura variety directly to the unitary group. One of us (M.H.) has written a fair number of papers about the Shimura varieties attached to similitude groups and is quite embarrassed to confess that he only learned of this possibility in 2017, after having failed to read carefully Chapter 27 of [GGP], and thus contributed significantly to the proliferation of excessively long formulas. Langlands's conjecture on the zeta functions of Shimura varieties ([L79]; see also Conjecture 0.4 below), applied in this case, yields a simple relation between the Galois representations on the ℓ-adic cohomology of the Shimura varieties attached to unitary groups, on the one hand, and the automorphic representations of $GL(n)$, on the other hand. However, these varieties are of *abelian type* but not of PEL type – they do not parametrize families of abelian varieties, but rather of motives closely related to those of abelian varieties. The proof of the Langlands conjecture – at places of good reduction – for Shimura varieties of abelian type, by the Langlands-Kottwitz method, is the subject of work in progress by Kisin, Shin, and Zhu [KSZ] that should appear in the near future.

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B. Geometry of Shimura varieties in positive characteristic

For arithmetic applications, the existence and characterization of *integral* canonical models for Shimura varieties play an important role. Let us discuss this problem in a fairly general context, where the Shimura variety $Sh(G, X)_{K_f}$ is associated to Shimura data such that $G_{\mathbb{Q}_p}$ is an unramified group and K_f factorizes as $K_f = K^p K_p$ for $K_p \subset G(\mathbb{Q}_p)$ a hyperspecial maximal compact subgroup and for $K^p \subset G(\mathbb{A}_f^p)$ a sufficiently small compact open subgroup. We fix K_p but let K^p vary, and consider the pro-E-scheme

$$Sh_{K_p}(G, X) = \varprojlim_{K^p} Sh(G, X)_{K^p K_p}.$$

Here $Sh(G, X)_{K^p K_p}$ denotes the canonical model of the Shimura variety over the reflex field E. An integral model of $Sh_{K_p}(G, X)$ over $\mathcal{O}_\mathfrak{p}$ consists of a $G(\mathbb{A}_f^p)$-equivariant inverse system $(\mathcal{S}_{K^p})$ of $\mathcal{O}_\mathfrak{p}$-models of the inverse system $(Sh(G, X)_{K^p K_p, E_\mathfrak{p}})_{K^p}$. We also view the pro-$\mathcal{O}_\mathfrak{p}$-scheme $\mathcal{S}_{K_p} = \varprojlim_{K^p} \mathcal{S}_{K^p}$ as an integral model of $Sh_{K_p}(G, X)_{E_\mathfrak{p}}$. We say such an integral model is *smooth* if for some subgroup K_0^p, $\mathcal{S}_{K_0^p}$ is smooth over $\mathcal{O}_\mathfrak{p}$ and $\mathcal{S}_{K_1^p}$ is étale over $\mathcal{S}_{K_2^p}$ for all $K_1^p \subseteq K_2^p \subseteq K_0^p$. Given our assumption on K_p, a smooth integral model should always exist. In the case where the Shimura data is PEL type (assuming $p > 2$ in Type D), Kottwitz [K92a] showed that the natural PEL moduli problem attached to $(G, X, K^p K_p)$ is represented by a smooth quasi-projective scheme over $\mathcal{O}_\mathfrak{p}$, essentially by reducing to the Siegel case $(\mathrm{GSp}, S^\pm)$, which was handled earlier by Mumford using geometric invariant theory [Mu65].

If $Sh_{K_p}(G, X)$ has at least one smooth integral model, in principle it will have many. So it is important to define a favorable notion of smooth integral model and characterize it by a condition which pins it down up to unique isomorphism. One such characterization, reminiscent of the Néron extension property of Néron models, was suggested by Milne [Mi92, §2]: the pro-$\mathcal{O}_\mathfrak{p}$-scheme $\mathcal{S}_{K_p}$ should be termed an *integral canonical model* if it is smooth (hence regular as a scheme) and satisfies the following extension property: given any regular $\mathcal{O}_\mathfrak{p}$-scheme Y such that $Y_{E_\mathfrak{p}}$ is dense in Y, any $E_\mathfrak{p}$-morphism $Y_{E_\mathfrak{p}} \to Sh_{K_p}(G, X)_{E_\mathfrak{p}}$ extends uniquely to an $\mathcal{O}_\mathfrak{p}$-morphism $Y \to \mathcal{S}_{K_p}$. As explained by Moonen [Mo98, §3], it is not clear such models always exist; it is not even clear that the Siegel modular scheme satisfies this extension property. Fundamentally, the problem is that the class of test schemes Y in this definition is too broad. In showing that the Siegel modular scheme satisfies Milne's extension property, one needs to know that for any closed subscheme $Z \subset Y$ disjoint from $Y_{E_\mathfrak{p}}$ and of codimension at least 2, any abelian scheme over $Y \backslash Z$ extends to an abelian scheme over all of Y. However, this statement is false in general ([Mo98, §3.4]). But the required version of it beomes true, thanks to a lemma of Faltings [Mo98, 3.6], if we require Y to also be *formally smooth* over $\mathcal{O}_\mathfrak{p}$ (see [Mo98, §3.5]). The problem with Milne's suggestion is therefore averted by modifying the extension property: we require it only of *regular* and *formally smooth* $\mathcal{O}_\mathfrak{p}$-schemes Y (see [Ki10, §2.3.7]). Then Kisin's main result in [Ki10] is that, for $p > 2$, every abelian type Shimura variety $Sh_{K_p}(G, X)$ possesses an integral canonical model $\mathcal{S}_{K_p}$ in this modified sense. Since $\mathcal{S}_{K_p}$ is itself regular and formally smooth over $\mathcal{O}_\mathfrak{p}$ by construction, the modified extension property characterizes $\mathcal{S}_{K_p}$ uniquely up to unique isomorphism. Vasiu proved results similar to Kisin's in his earlier works [Va99],

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but the characterization is stated in a different way. It is not hard to show that Kottwitz's PEL moduli problems are represented by (finite unions of) integral canonical models. In particular this applies to the PEL unitary Shimura varieties which are the main focus of this volume.

Chapter 5 is a summary of Kisin's paper [Ki10], which constructs integral canonical models for Shimura varieties of abelian type (assuming $p > 2$) according to the following outline:

(i) Hodge type case

For Hodge type data (G, X) endowed with an embedding of Shimura data $(G, X) \hookrightarrow (\mathrm{GSp}(V), S^{\pm})$, one constructs an integral canonical model S_{K_p} as the normalization of the closure of $Sh_{K_p}(G, X)$ in $S_{K'_p}(\mathrm{GSp}(V), S^{\pm})$ for $K'_p \subset \mathrm{GSp}(\mathbb{Q}_p)$ the stabilizer of a certain lattice $V_{\mathbb{Z}} \subset V$. Here, the integral model $S_{K'_p}(\mathrm{GSp}(V), S^{\pm})$ comes from a moduli problem, formulated with the choice of $V_{\mathbb{Z}}$. (Warning: the latter is not necessarily an integral canonical model associated to $(\mathrm{GSp}(V), S^{\pm})$ so in particular is not necessarily the same as Kottwitz' Siegel modular scheme, unless K'_p is hyperspecial.)

This first step relies on Kisin's Key Lemma, which asserts that certain tensors arising from certain Hodge classes via the p-adic comparision theorems are *integral*. For this Kisin uses his earlier results on crystalline representations. Then Kisin is able to make use of Faltings' deformation ring for p-divisible groups equipped with a collection of Tate cycles [Fa99, §7] to show that S_{K_p} as constructed above is smooth and satisfies the modified extension property.

(ii) Abelian type case

Let (G, X) be a Shimura variety of abelian type, and let (G_1, X_1) be a Shimura variety of Hodge type such that there is a central isogeny $G_1^{der} \to G^{der}$ inducing an isomorphism $(G_1^{ad}, X_1^{ad}) \xrightarrow{\sim} (G^{ad}, X^{ad})$. Kisin constructs the integral canonical model for $Sh_{K_p}(G, X)$ from the one already constructed for $Sh_{K_{1,p}}(G_1, X_1)$.

Having the integral canonical model S_{K_p} in hand, it now makes sense to consider its special fiber $\bar{S}_{K_p}$, which is a well-defined object depending canonically on the data (G, X, K_p). Again, for PEL type Shimura varieties like the ones we consider in this volume, this amounts to studying the Kottwitz moduli problems over the residue field of $\mathcal{O}_{\mathfrak{p}}$. Even in PEL situations, there are difficult questions revolving around the counting of points (used to understand zeta functions and to construct Galois representations in the cohomology of Shimura varieties), and also questions related to various stratifications (Newton, Ekedahl-Oort, Kottwitz-Rapoport, etc.). In this volume, the point-counting problems and their relations to automorphic forms and Galois representations are addressed in the articles (Zhu), (Shin), and (Scholze), and their contents are further described in Parts A and C of this introduction. In the rest of Part B, we discuss the articles (Mantovan) and (Viehmann), which are concerned with the geometry of the special fibers $\bar{S}_{K_p}$, most importantly with the Newton stratification, the Oort foliations, and their applications. Although some statements are currently known to extend to Hodge type Shimura varieties, for the most part we will limit our discussion to PEL Shimura varieties, for simplicity.

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Let $k \cong \mathbb{F}_p$ be a finite prime field, and let L denote the fraction field of the Witt ring $W(\bar{k})$. Let σ be the Frobenius automorphism of $W(\bar{k})$ over $W(k)$ induced by $x \mapsto x^p$ on k, and use the same symbol for the automorphism of L. An F-*isocrystal* is a pair (V, Φ) consisting of a finite-dimensional L-vector space V and a σ-linear bijection $\Phi : V \to V$.

Using the Dieudonné-Manin classification of F-isocrystals, one can associate to a simple F-isocrystal its Newton polygon. In [Ko85], Kottwitz defined, for any connected reductive group $G/\mathbb{Q}_p$, the set of F-isocrystals with G-structure (called G-*isocrystals* here), and showed that when G is quasi-split this set is identified with the pointed set $B(G)$ consisting of σ-conjugacy classes in $G(L)$. In the same setting he gave a group-theoretic meaning to Newton polygons, defining the set of *Newton points*

$$\mathcal{N}(G) = (X_*(A)_{\mathbb{Q}}/\Omega)^{\Gamma}$$

where A is a maximal $\mathbb{Q}_p$-torus with Weyl group Ω and $\Gamma = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$. Further, Kottwitz gave a group-theoretic generalization of the map to Newton polygons, by defining the *Newton map*

$$\bar{\nu} \; : \; B(G) \to \mathcal{N}(G) \; , \; \; b \mapsto \bar{\nu}_b$$

(see (Viehmann), §3). Here $\nu_b \in \mathrm{Hom}_L(\mathbb{D}, G)$ refers to the *slope homomorphism* which is the group-theoretic counterpart of slopes of an isocrystal (see (Mantovan), §2.3). In the cases of interest to us (where $G^{der} = G^{sc}$ and G is quasisplit over $\mathbb{Q}_p$), the map $\bar{\nu}$ is injective, and the natural partial order $\preceq$ on $\mathcal{N}(G)$ gives $B(G)$ the structure of a poset.

Now return to the global data (G, X), where $G_{\mathbb{Q}_p}$ is still assumed to be quasi-split, and let $\mu_{\bar{\mathbb{Q}}_p}$ denote a geometric cocharacter of $G_{\bar{\mathbb{Q}}_p}$ coming from (G, X) in the usual way. Kottwitz defined a finite subset $B(G_{\mathbb{Q}_p}, \mu_{\bar{\mathbb{Q}}_p}) \subset B(G)$ using $\preceq$. In PEL cases, a closed geometric point x of $\bar{S}_{K_p}$ gives rise to the $G_{\mathbb{Q}_p}$-isocrystal associated to the (rational) Dieudonné module $\mathbb{D}H_{x,\mathbb{Q}}$ of its associated p-divisible group H_x, and thus to an element $[b_x] \in B(G_{\mathbb{Q}_p}, \mu_{\bar{\mathbb{Q}}_p})$. The *Newton stratum* (resp. closed Newton set) associated to $[b] \in B(G_{\mathbb{Q}_p}, \mu_{\bar{\mathbb{Q}}_p})$ is the locus $\mathcal{N}_{[b]}$ (resp. $\mathcal{N}_{\preceq[b]}$) of points $x \in \bar{S}_{K_p}(\bar{k})$ such that $[b_x] = [b]$ (resp. $[b_x] \preceq [b]$). It is known thanks to Grothendieck (for p-divisible groups without additional structure) and thanks to Rapoport-Richartz [RR96] in general, that $\mathcal{N}_{\preceq[b]}$ is indeed a closed subset.

Fundamental quesions one can ask about Newton strata for $[b] \in B(G_{\mathbb{Q}_p}, \mu_{\bar{\mathbb{Q}}_p})$:

- Is every Newton stratum $\mathcal{N}_{[b]}$ non-empty?
- Does the stratification behave well, i.e., is $\mathcal{N}_{\preceq[b]}$ the union of the $\mathcal{N}_{[b']}$ for $[b'] \preceq [b]$?
- What is the geometry of $\mathcal{N}_{[b]}$, in particular, what is its dimension?
- How does $\mathcal{N}_{[b]}$ relate to other important objects, such as Rapoport-Zink spaces $RZ_{(G,b,\mu)}$ and the central leaves $C_{[b]}$ in $\mathcal{N}_{[b]}$? (See (Mantovan), §3, 4.)

A large part of the article (Viehmann) is a summary of the current knowledge about these questions. The first point has been established in many cases, including in the PEL cases of interest to us, by Viehmann-Wedhorn, see Theorem 4.1 of chapter (Viehmann); in addition (Viehmann) explains various others approaches to the non-emptiness question. The second

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point was proved, in both PEL and Hodge type cases, by Hamacher [Ha15b, Ha17] (see (Viehmann), Theorem 5.2). The most important part of the proof in the PEL case is to first answer the third point, in particular to prove that $\mathcal{N}_{[b]}$ is equidimensional and that $\dim \mathcal{N}_{[b]}$ is as conjectured (up to a minor correction) by Rapoport [Ra05, p. 296] – see (Viehmann), Theorem 5.7. An essential tool is the relation

$$\dim \mathcal{N}_{[b]} = \dim X_\mu(b)_{\mathbb{Q}_p} + \dim C_{[b]}, \tag{0.1}$$

where $C_{[b]}$ is the *central leaf* in the Newton stratum and $X_\mu(b)_{\mathbb{Q}_p}$ is the *affine Deligne-Lusztig variety*, which can be identified with the reduced Rapoport-Zink space $\overline{RZ}_{(G,b,\mu)}$. The equation (0.1) reflects the Oort-Mantovan structure of $\mathcal{N}_{[b]}$ as an "quasi-product" of $RZ_{(G,b,\mu)}$ and $C_{[b]}$; see (Mantovan), §5.

The affine Deligne-Lusztig variety has a counterpart in the equal characteristic world, $X_\mu(b)_{\mathbb{F}_p((t))}$, which had been intensively studied by Görtz-Haines-Kottwitz-Reuman [GHKR], Viehmann [Vi06] (for split groups G over $\mathbb{F}_p((t))$) and by Hamacher [Ha15a] (for unramified groups), and also by many others. In particular the cited works completely proved Rapoport's conjectural closed formula for $\dim X_\mu(b)$, when G is unramified but μ is arbitrary (not necessarily minuscule) – see (Viehmann) Theorem 5.13. In [Ha15b, Ha17], Hamacher transported the techniques of these papers over to the p-adic context $X_\mu(b)_{\mathbb{Q}_p}$, and deduced the required dimension formulae for PEL and Hodge type Shimura varieties.

The quasi-product structure of $\mathcal{N}_{[b]}$ is discussed in greater depth in (Mantovan), §5, and is extended to quasi-products of Igusa varieties $\mathrm{Ig}_{m,\mathbb{X}}$ and elements in the truncated Rapoport-Zink tower, denoted in (Mantovan) by $\mathcal{M}^{n,d}_{b,\mathbb{X}}$, where $\mathbb{X}$ is a certain "completely slope divisible" p-divisible group compatible in a certain sense with $\mu_{\bar{\mathbb{Q}}_p}$. The quasi-product structure refers to finite surjective morphisms

$$\mathrm{Ig}_{m,\mathbb{X}} \times \bar{\mathcal{M}}^{n,d}_{b,\mathbb{X}} \to \mathcal{N}_{[b]}(\bar{k}).$$

These structures give rise to Mantovan's formula expressing the cohomology of $\mathcal{N}_{[b]}$ in terms of the cohomology of the corresponding Igusa varieties and Rapoport-Zink spaces, see (Mantovan), §5.3.

Finally, we mention that all of the above questions can be framed and studied for more general subgroups K_p. In particular, the cases where K_p is a parahoric subgroup have attracted a lot of attention in recent years, but unfortunately this is beyond the scope of the present volume and we will not attempt to summarize the progress that has been made in this direction. Let us only mention that Kisin and Pappas [KP] have recently succeeded in constructing integral models S_{K_p} for abelian type Shimura varieties $Sh_{K_p}(G, X)$ in most cases where K_p is parahoric and $p > 2$. Furthermore, it is fully expected that these integral models, while neither regular nor formally smooth, are nevertheless characterized uniquely by a kind of valuative criterion of properness for characteristic $(0, p)$ valuation rings. A careful study of the fine structure of their reductions modulo p has borne some fruit but much work remains to be done. The reader is encouraged to consult the paper of He and Rapoport [HeRa] for a contemporary (group-theoretic) point of view unifying all the various stratifications which are considered in this subject.

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