Introduction

Gauss described number theory as the queen of mathematics. Indeed, the amount of mathematics invented for arithmetic reasons is truly astonishing. To name just a few examples:

- A large part of complex analysis (Cauchy, Riemann, Weierstraß, Hadamard, Hardy–Littlewood, . . .);
- The theory of divisors (Kummer, Dedekind) and of ideals (E. Noether);
- The theory of Riemann surfaces (Riemann, . . .);
- The non-analytic version of the Riemann–Roch theorem for curves (F. K. Schmidt);
- The refoundation of Italian algebraic geometry on the basis of commutative algebra (Weil);[1]
- The theory of abelian varieties (Weil);
- Part of the theory of linear representations of finite groups (Artin, Brauer);
- A large part of homological algebra: group cohomology, the theory of sheaves on a general site (Cartan, Eilenberg, Serre, Tate, Grothendieck, . . .);
- Part of the theory of schemes (Grothendieck);
- The development of étale cohomology (M. Artin, Grothendieck, Verdier, Deligne. . .) and then crystalline cohomology (Grothendieck, Berthelot, Ogus, Bloch, Deligne, Illusie, . . .);
- Part of the theory of derived categories (Grothendieck, Verdier);
- The six operations formalism (Grothendieck);
- Monodromy theory in the world of schemes (Grothendieck, Serre, Deligne, Katz, . . .);

and, of course, the theory of motives!

[1] van der Waerden and Zariski also participated in this movement, for different reasons.

Zeta and L-functions are the meeting point of these theories: they crown the queen of mathematics. It is remarkable that a function whose definition is as simple as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

has played such a deep role, admitting vast generalisations which have shaped the evolution of number theory up to the present day; yet it remains the object of a conjecture whose analogue for varieties over finite fields was famously proved by Weil and Deligne, but whose original case due to Riemann remains an unapproachable mystery.

The text that follows is derived from a master's level course given at Jussieu in 2013. I have tried to give an exposition of part of the known results on zeta and L-functions, but also of the complex issues surrounding them, in an *ontogenetic* way: ontogenesis describes the progressive development of an organism from its conception to its mature form. With this goal in mind, I have scattered the text with quotations and commentaries which, I hope, will offer the reader a small window on the history of ideas in this field.

After having recalled in Chapter 1 the classical results (and hypothesis) on the Riemann zeta function, I introduce zeta functions of $\mathbf{Z}$-schemes of finite type in the second chapter, which is essentially dedicated to the proof of the Riemann hypothesis for curves over a finite field. To my great regret, I was not able to do justice to Weil's proof of the Castelnuovo–Severi inequality in [Wei2]: to re-transcribe it in the language of schemes would have led me too far astray (see [64]). I only include in § 2.9.5 an idea of the proof, and treat the easy case of curves of genus 1 (due to Hasse). For the general case, I followed everyone else in giving the proofs of Mattuck–Tate and Grothendieck, which rely on an *a priori* weaker inequality; as a consolation prize, I compare the two inequalities in § 2.9.7 and show that we can recover the first one using the second and the additivity of Pic^{τ} (the numerically trivial divisors). A second proof of the Castelnuovo–Severi inequality via abelian varieties will be found in the next chapter (§ 3.2.2). It explains the first proof and puts it in a new light.

Chapter 3 is dedicated to the Weil conjectures. They are all proven, except for the hardest one: the "Riemann hypothesis". I also give an overview of Dwork's p-adic proof of the rationality of zeta functions of varieties over a finite field (obtained before the development of Grothendieck's cohomological methods!).

Chapter 4 returns to more elementary mathematics, introducing Dirichlet, Hecke, and Artin L-functions. I give a proof of Dirichlet's theorem on arithmetic progressions, by the method expounded by Serre in [Ser3]; it would however be a shame, as Pierre Charollois remarked to me, to omit Dirichlet's original method, which gave additional information and anticipated the analytic class number formulae (see Remark 4.22(2)). I then introduce the two generalisations of Dirichlet's L-functions: those of Hecke and Artin. I state without proof Hecke's main theorem: existence of an analytic continuation and a functional equation (Theorem 4.60), and then explain how Artin and Brauer derived the same results for non-abelian L-functions (Theorem 4.69).

The next chapter is the *pièce de résistance* of the text. It starts by introducing the approximate idea of Hasse–Weil L-functions, and ends by describing their precise definition due to Serre [113]. In the mean time, I explain the fundamental contributions of Grothendieck and Deligne: rationality and the functional equation for L-functions of l-adic sheaves in characteristic p, with essentially complete proofs; the theory of weights; and some theorems of Deligne on the Riemann hypothesis (the last of Weil's conjectures), this time without proof. Notably, one will find in § 5.4.4 an exposition of functional equations of the L-functions developed by Grothendieck in [CorrGS, letter from 30-9-64], and one will find in Theorem 5.58 a more precise statement and fairly complete parts of the proof of Grothendieck and Deligne's theorem on the rationality and the functional equation of Hasse–Weil L-functions in characteristic > 0, confirming a conjecture of Serre (Conjecture 5.57) in this case.

The last chapter is dedicated to motives and their zeta functions. I limited myself to an elementary case: that of pure motives of Grothendieck, associated to smooth projective varieties over a finite field. One can go much further using the triangulated categories of motives introduced by Voevodsky and developed by Ivorra, Ayoub, and Cisinski–Déglise, but this would go beyond the scope of the book (see [63]). I explain how this viewpoint considerably clarifies how Weil cohomologies are used to prove rationality and the functional equation, and I do not resist the pleasure of applying this theory to prove a somewhat forgotten theorem of Weil: Artin's conjecture for non-abelian L-functions in positive characteristic (§ 6.15).

Finally, two appendices give supplements from categorical algebra. I also scattered the text with exercises, without attempting anything systematic.

Aside from Hecke theory, the automorphic side of the story is left essentially unaddressed; I contented myself with brief allusions here and there.

There still remains one question: what is the difference between a zeta function and an L-function? Morally, an L-function is a zeta function "with coefficients" (in a sheaf, in a representation, ...), and we recover a zeta function by taking trivial coefficients (cf. § 5.3). The situation becomes more complicated when we consider the Euler products associated by Hasse and Weil to a curve over a number field... Another problem is which terminology one should use in the case of motives. For pure motives, I have chosen that of zeta functions, which seems well established. It would probably be simpler to unify the terminology by dropping one of the two notations, but this is obviously delicate, for reasons of tradition.

I have benefited a great deal from previous expositions of the theory, from which I have borrowed much: these expositions would be too numerous to list here comprehensively, so I simply refer the reader to the bibliography. I thank the audience from the course, and in particular Matthieu Rambaud, for pointing out a number of misprints. Finally, I thank Joseph Ayoub, Pierre Charollois, Luc Illusie, Amnon Neeman, Ram Murty, and Jean-Pierre Serre for pertinent comments on this text.

Guide to the reader

This book is not a systematic exposition of a theory of zeta and L-functions – which does not exist, contrary to class field theory for example. I have chosen primarily to highlight the history of ideas. Thus, while certain proofs are complete, in others I prefer to emphasise the main ideas rather than give a complete but excessively technical proof; when such a proof is already available, I refer to expositions containing the missing parts.

Finally, there are cases where I have provided details that cannot be found in the literature. Here are some of them:

- § 5.2: a discussion of the notion of good reduction in relation to Hasse–Weil zeta functions (see in particular Proposition 5.4).
- § 5.4: Grothendieck's functional equation, described in [CorrGS, letter from 30-9-64] (Theorems 5.29 and 5.30).
- Theorem 5.58, deducing from this functional equation the one described by Serre in [113, 4.1, example a)], with a precision that follows from "Weil II" (Deligne [30]).
- § 6.8: the theory of specialisation for pure motives.
- §§ 6.12 and 6.13: the zeta function of an endomorphism of a motive, its rationality, and its functional equation (see also § A.2.8 for an abstract categorical version, reproducing that of [60, I] in a slightly weaker form).
- § 6.15: a motivic proof of Artin's conjecture in positive characteristic.
- § A.2.4: the notion of a "category with suspension".

I have also recalled (without proof) several standard results, such as the Riemann–Roch theorem, the theory of abelian varieties, and the basics of algebraic number theory, for which there are already excellent expositions.

Finally, I assume some prerequisites, varying from one chapter to another:

- Chapters 1, 2 and §§ 4.1, 4.2: the basics of complex analysis.
- § 1.10 and Chapter 4: the basics of algebraic number theory, they can be found for example in the book of Lang [Lan2], in Cassels–Fröhlich [CF], among others.
- Chapters 2 and 3: the basics of algebraic geometry, as can be found for example in the book of Hartshorne [Har2]; the basics of intersection theory, as can be found in the book of Fulton [Ful].
- § 4.3: the basics of general topology.
- §§ 4.4 and 4.6: the basics of the theory of linear representations of finite groups, as in Serre's book [Ser1].
- Chapter 5: much more serious foundations of the theory of schemes and étale cohomology, as developed in [SGA4, SGA4$\frac{1}{2}$, SGA5] and [Mil]. This chapter is undoubtedly the hardest to read.
- Chapter 6: the basics of categorical algebra, as in the book of Mac Lane [Mcl].

Preface to the English edition

This version is a direct translation of the French version; it differs only by a few corrections and the addition of four exercises in Chapter 6. The numbering has also been changed.