Cambridge University Press
978-1-108-56294-2 — Cambridge International AS & A Level Mathematics Mechanics Coursebook with Cambridge Online Mathematics (2 Years)
Jan Dangerfield , Stuart Haring , Edited by Julian Gilbey
Excerpt
More Information

Chapter 1
Velocity and acceleration

In this chapter you will learn how to:

- work with scalar and vector quantities for distance and speed
- use equations of constant acceleration
- sketch and read displacement–time graphs and velocity–time graphs
- solve problems with multiple stages of motion.

PREREQUISITE KNOWLEDGE

Where it comes from	What you should be able to do	Check your skills
IGCSE® / O Level Mathematics	Solve quadratics by factorising or using the quadratic formula.	**1** Solve the following equations. **a** $x^2 - 2x - 15 = 0$ **b** $2x^2 + x - 3 = 0$ **c** $3x^2 - 5x - 7 = 0$
IGCSE / O Level Mathematics	Solve linear simultaneous equations.	**2** Solve the following pairs of simultaneous equations. **a** $2x + 3y = 8$ and $5x - 2y = 1$ **b** $3x + 2y = 9$ and $y = 4x - 1$

What is Mechanics about?

How far should the driver of a car stay behind another car to be able to stop safely in an emergency? How long should the fuse on a firework be so the firework goes off at the highest point? How quickly should you roll a ball so it stops as near as possible to a target? How strong does a building have to be to survive a hurricane? Mechanics is the study of questions such as these. By modelling situations mathematically and making suitable assumptions you can find answers to these questions.

In this chapter, you will study the motion of objects and learn how to work out where an object is and how it is moving at different times. This area of Mechanics is known as 'dynamics'. Solving problems with objects that do not move is called 'statics'; you will study this later in the course.

1.1 Displacement and velocity

An old English nursery rhyme goes like this:

The Grand Old Duke of York,
He had ten thousand men,
He marched them up to the top of the hill,
And he marched them down again.

His men had clearly marched some distance, but they ended up exactly where they started, so you cannot work out how far they travelled simply by measuring how far their finishing point is from their starting point.

You can use two different measures when thinking about how far something has travelled. These are **distance** and **displacement**.

Distance is a **scalar** quantity and is used to measure the total length of path travelled. In the rhyme, if the distance covered up the hill were 100 m, the total distance in marching up the hill and then down again would be $100\,\text{m} + 100\,\text{m} = 200\,\text{m}$.

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Displacement is a **vector** quantity and gives the location of an object relative to a fixed reference point or **origin**. In this course, you will be considering dynamics problems in only one dimension. To define the displacement you need to define one direction as positive. In the rhyme, if you take the origin to be the bottom of the hill and the positive direction to be up the hill, then the displacement at the end is 0 m, since the men are in the same location as they started. You can also reach this answer through a calculation. If you assume that they are marching in a straight line, then marching up the hill is an increase in displacement and marching down the hill is a decrease in displacement, so the total displacement is $(+100\,\text{m}) + (-100\,\text{m}) = 0\,\text{m}$.

Since you will be working in only one dimension, you will often refer to the displacement as just a number, with positive meaning a displacement in one direction from the origin and negative meaning a displacement in the other direction. Sometimes the direction and origin will be stated in the problem. In other cases, you will need to choose these yourself. In many cases the origin will simply be the starting position of an object and the positive direction will be the direction the object is moving initially.

TIP

A scalar quantity, such as distance, has only a magnitude. A vector quantity, such as displacement, has magnitude and direction. When you are asked for a vector quantity such as displacement or velocity, make sure you state the direction as well as the magnitude.

KEY POINT 1.1

Displacement is a measure of location from a fixed origin or starting point. It is a vector and so has both magnitude and direction. If you take displacement in a given direction to be positive, then displacement in the opposite direction is negative.

WEB LINK

Try the *Discussing distance* resource at the *Introducing calculus* station on the Underground Mathematics website (www.undergroundmathematics.org).

We also have two ways to measure how quickly an object is moving: **speed** and **velocity**. Speed is a scalar quantity, so has only a magnitude. Velocity is a vector quantity, so has both magnitude and direction.

For an object moving at constant speed, if you know the distance travelled in a given time you can work out the speed of the object.

KEY POINT 1.2

For an object moving at constant speed:

$$\text{speed} = \frac{\text{distance covered}}{\text{time taken}}$$

This is valid only for objects moving at constant speed. For objects moving at non-constant speed you can consider the average speed.

KEY POINT 1.3

$$\text{average speed} = \frac{\text{total distance covered}}{\text{total time taken}}$$

Velocity measures how quickly the displacement of an object changes. You can write an equation similar to the one for speed.

KEY POINT 1.4

For an object moving at constant velocity:

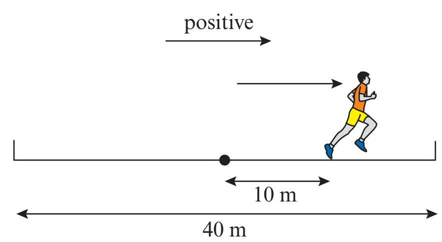
$$\text{velocity} = \frac{\text{change in displacement}}{\text{time taken}}$$

Let's see what this means in practice.

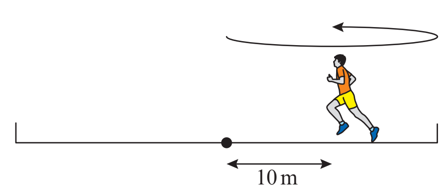
Suppose a man is doing a fitness test. In each stage of the test he runs backwards and forwards along the length of a small football pitch. He starts at the centre spot, runs to one end of the pitch, changes direction and runs to the other end, changes direction and runs back to the centre spot, as shown in the diagrams. He runs at $4\,\text{m}\,\text{s}^{-1}$ and the pitch is 40 m long.

To define displacement and velocity you will need to define the origin and the direction you will call positive. Let's call the centre spot the origin and to the right as positive.

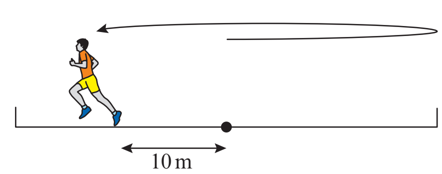
In the first diagram, he has travelled a distance of 10 m. Because he is 10 m in the positive direction, his displacement is 10 m. His speed is $4\,\text{m}\,\text{s}^{-1}$. Because he is moving in the positive direction, his velocity is also $4\,\text{m}\,\text{s}^{-1}$.



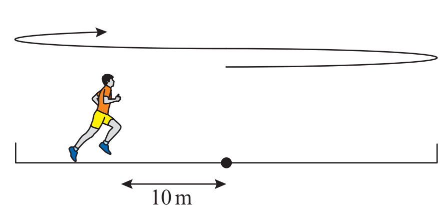
In the second diagram, he has travelled a total distance of 30 m, but he is only 10 m from the centre spot, so his displacement is 10 m. His speed is still $4\,\text{m}\,\text{s}^{-1}$ but he is moving in the negative direction so his velocity is $-4\,\text{m}\,\text{s}^{-1}$.



In the third diagram, he has travelled a total distance of 50 m, but he is now 10 m from the centre spot in the negative direction, so his displacement is -10 m. His speed is still $4\,\text{m}\,\text{s}^{-1}$ and he is still moving in the negative direction so his velocity is still $-4\,\text{m}\,\text{s}^{-1}$.



In the fourth diagram, he has travelled a total distance of 70 m, but his displacement is still -10 m. His speed is still $4\,\text{m}\,\text{s}^{-1}$ and he is moving in the positive direction again so his velocity is also $4\,\text{m}\,\text{s}^{-1}$.



The magnitude of the velocity of an object is its speed. Speed can never be negative. For example, an object moving with a velocity of $+10\,\text{m}\,\text{s}^{-1}$ and an object moving with a velocity of $-10\,\text{m}\,\text{s}^{-1}$ both have a speed of $10\,\text{m}\,\text{s}^{-1}$.

TIP

We use vertical lines to indicate magnitude of a vector.

So, speed $= |\text{velocity}|$

As with speed, for objects moving at non-constant velocity you can consider the average velocity.

KEY POINT 1.5

$$\text{average velocity} = \frac{\text{net change in displacement}}{\text{total time taken}}$$

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In the previous example, the man's average speed is $4\,\mathrm{m\,s^{-1}}$ but his average velocity is $0\,\mathrm{m\,s^{-1}}$.

We can rearrange the equation for velocity to deduce that for an object moving at constant velocity v for time t, the change in displacement s (in the same direction as the velocity) is given by:

$$s = vt$$

The standard units used for distance and displacement are metres (m) and for time are seconds (s). Therefore, the units for speed and velocity are metres per second (usually written in mathematics and science as $\mathrm{m\,s^{-1}}$, although you may also come across the notation m/s). These units are those specified by the *Système Internationale* (SI), which defines the system of units used by scientists all over the world. Other commonly used units for speed include kilometres per hour (km/h) and miles per hour (mph).

WEB LINK

Try the *Speed vs velocity* resource at the *Introducing calculus* station on the Underground Mathematics website.

WORKED EXAMPLE 1.1

A car travels 9 km in 15 minutes at constant speed. Find its speed in $\mathrm{m\,s^{-1}}$.

Answer

$9\,\mathrm{km} = 9000\,\mathrm{m}$ and
$15\,\text{minutes} = 900\,\mathrm{s}$

Convert to units required for the answer, which are SI units.

$s = vt$
so $9000 = 900v$
$v = 10\,\mathrm{m\,s^{-1}}$

Substitute into the equation for displacement and solve.

WORKED EXAMPLE 1.2

A cyclist travels at $5\,\mathrm{m\,s^{-1}}$ for 30 s then turns back, travelling at $3\,\mathrm{m\,s^{-1}}$ for 10 s. Find her displacement in the original direction of motion from her starting position.

Answer

$s = vt$
So $s_1 = 5 \times 30$
$= 150$

Separate the two stages of the journey.

and $s_2 = -3 \times 10$
$= -30$

Remember travelling back means a negative velocity and a negative displacement.

So the total displacement is
$s = 150 + (-30)$
$= 120\,\mathrm{m}$

TIP

You usually only include units in the final answer to a problem and not in all the earlier steps. This is because it is easy to confuse units and variables. For example, s for displacement can be easily mixed up with s for seconds. It is important to work in SI units throughout, so that the units are consistent.

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WORKED EXAMPLE 1.3

A cyclist spends some of his journey going downhill at $15\,\mathrm{m\,s^{-1}}$ and the rest of the time going uphill at $5\,\mathrm{m\,s^{-1}}$. In 1 minute he travels 540 m. Find how long he spent going downhill.

Answer

Let t be the amount of time spent going downhill. — Define the variable.

Then $60 - t$ is the amount of time spent going uphill. — Write an expression for the time spent travelling uphill.

Total distance $= 15t + 5(60 - t) = 540$ — Set up an equation for the total distance.

$$15t + 300 - 5t = 540$$
$$10t = 240$$
$$t = 24\,\mathrm{s}$$

EXPLORE 1.1

Two students are trying to solve this puzzle.

A cyclist cycles from home uphill to the shop at $5\,\mathrm{m\,s^{-1}}$. He then cycles home and wants to average $10\,\mathrm{m\,s^{-1}}$ for the total journey. How fast must he cycle on the way home?

The students' solutions are shown here. Decide whose logic is correct and try to explain what is wrong with the other's answer.

Student A	Student B
Call the speed on the return journey v. The average of $5\,\mathrm{m\,s^{-1}}$ and v is $10\,\mathrm{m\,s^{-1}}$, so v must be $15\,\mathrm{m\,s^{-1}}$.	Cycling at $5\,\mathrm{m\,s^{-1}}$ will take twice as long as it would if he were going at $10\,\mathrm{m\,s^{-1}}$. That means he has used up the time required to go there and back in the first part of the journey, so it is impossible to average $10\,\mathrm{m\,s^{-1}}$ for the total journey.

WEB LINK

You may want to have a go at the *Average speed* resource at the *Introducing calculus* station on the Underground Mathematics website.

MODELLING ASSUMPTIONS

Throughout this course, there will be questions about how realistic your answers are. To simplify problems you will make reasonable assumptions about the scenario to allow you to solve them to a satisfactory degree of accuracy. To improve the agreement of your model with what happens in the real world, you would need to refine your model, taking into account factors that you had initially ignored.

In some of the questions so far, you might ask if it is reasonable to assume constant speed. In real life, speed would always change slightly, but it could be close enough to constant that it is a reasonable assumption.

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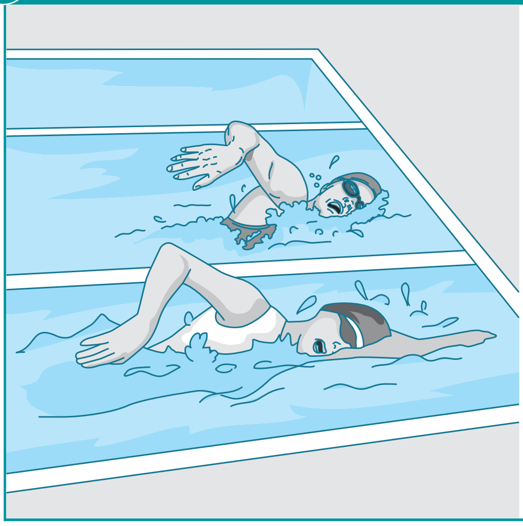
With real objects, such as bicycles or cars, there is the question of which part of the object you are referring to. You can be consistent and say it is the front of a vehicle, but when it is a person the front changes from the left leg to the right leg. You may choose to consider the position of the torso as the position of the person. In all the examples in this coursebook, you will consider the object to be a particle, which is very small, so you do not need to worry about these details. You will assume any resulting errors in the calculations will be sufficiently small to ignore. This could cause a problem when you consider the gap between objects, because you may not have allowed for the length of the object itself, but in our simple models you will ignore this issue too.

DID YOU KNOW?



Once they have reached top speed, swimmers tend to move at a fairly constant speed at all points during the stroke. However, the race ends when the swimmer touches the end of the pool, so it is important to time the last two or three strokes to finish with arms extended. If the stroke finishes early the swimmer might not do another stroke and instead keep their arms extended, but this means the swimmer slows down. In a close race, another swimmer may overtake if that swimmer times their strokes better. This happened to Michael Phelps when he lost to Chad Le Clos in the final of the Men's 200 m Butterfly in the 2012 London Olympics.

EXERCISE 1A

1 A cyclist covers 120 m in 15 s at constant speed. Find her speed.

2 A sprinter runs at constant speed of $9\,\text{m}\,\text{s}^{-1}$ for 7 s. Find the distance covered.

M 3 a A cheetah spots a grazing gazelle 150 m away and runs at a constant $25\,\text{m}\,\text{s}^{-1}$ to catch it. Find how long the cheetah takes to catch the gazelle.

b What assumptions have been made to answer the question?

4 The speed of light is $3.00 \times 10^8\,\text{m}\,\text{s}^{-1}$ to 3 significant figures. The average distance between the Earth and the Sun is 150 million km to 3 significant figures. Find how long it takes for light from the Sun to reach the Earth on average. Give the answer in minutes and seconds.

5 The land speed record was set in 1997 at $1223.657\,\text{km}\,\text{h}^{-1}$. Find how long in seconds it took to cover 1 km when the record was set.

M 6 A runner runs at $5\,\text{m}\,\text{s}^{-1}$ for 7 s before increasing the pace to $7\,\text{m}\,\text{s}^{-1}$ for the next 13 s.

a Find her average speed.

b What assumptions have been made to answer the question?

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7 A remote control car travels forwards at $6\,\text{m}\,\text{s}^{-1}$ in Drive and backwards at $3\,\text{m}\,\text{s}^{-1}$ in Reverse. The car travels for 10 s in Drive before travelling for 5 s in Reverse.

a Find its displacement from its starting point.

b Find its average velocity in the direction in which it started driving forwards.

c Find its average speed.

8 A speed skater averages $11\,\text{m}\,\text{s}^{-1}$ over the first 5 s of a race. Find the average speed required over the next 10 s to average $12\,\text{m}\,\text{s}^{-1}$ overall.

9 The speed of sound in wood is $3300\,\text{m}\,\text{s}^{-1}$ and the speed of sound in air is $330\,\text{m}\,\text{s}^{-1}$. A hammer hits one end of a 33 m long plank of wood. Find the difference in time between the sound waves being detected at the other end of the plank and the sound being heard through the air.

10 An exercise routine involves a mixture of jogging at $4\,\text{m}\,\text{s}^{-1}$ and sprinting at $7\,\text{m}\,\text{s}^{-1}$. An athlete covers 1 km in 3 minutes and 10 seconds. Find how long she spent sprinting.

11 Two cars are racing over the same distance. They start at the same time, but one finishes 8 s before the other. The faster one averaged $45\,\text{m}\,\text{s}^{-1}$ and the slower one averaged $44\,\text{m}\,\text{s}^{-1}$. Find the length of the race.

12 Two air hockey pucks are 2 m apart. One is struck and moves directly towards the other at $1.3\,\text{m}\,\text{s}^{-1}$. The other is struck 0.2 s later and moves directly towards the first at $1.7\,\text{m}\,\text{s}^{-1}$. Find how far the first puck has moved when the collision occurs and how long it has been moving for.

P 13 A motion from point A to point C is split into two parts. The motion from A to B has displacement s_1 and takes time t_1. The motion from B to C has displacement s_2 and takes time t_2.

a Prove that if $t_1 = t_2$, the average speed from A to C is the same as the average of the speeds from A to B and from B to C.

b Prove that if $s_1 = s_2$, the average speed from A to C is the same as the average of the speeds from A to B and from B to C if, and only if, $t_1 = t_2$.

P 14 The distance from point A to point B is s. In the motion from A to B and back, the speed for the first part of the motion is v_1 and the speed for the return part of the motion is v_2. The average speed for the entire motion is v.

a Prove that $v = \dfrac{2v_1v_2}{v_1 + v_2}$.

b Deduce that it is impossible to average twice the speed of the first part of the motion; that is, it is impossible to have $v = 2v_1$.

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1.2 Acceleration

Velocity is not the only measure of the motion of an object. It is useful to know if, and how, the velocity is changing. We use **acceleration** to measure how quickly velocity is changing.

KEY POINT 1.6

For an object moving at constant acceleration,

$$\text{acceleration} = \frac{\text{change in velocity}}{\text{time taken}}$$

If an object has constant acceleration a, initial velocity u and it reaches final velocity v in time t, then

$$a = \frac{v - u}{t}$$

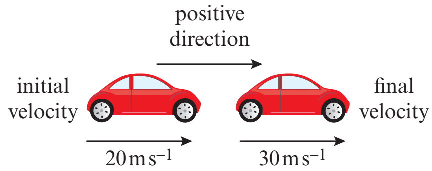
where u, v and a are all measured in the same direction.

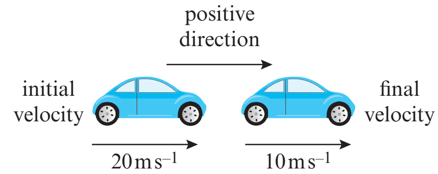
TIP

The units of acceleration are m s^{-2}.

An increase in velocity is a positive acceleration, as shown in the diagram on the left.

A decrease in velocity is a negative acceleration, as shown in the diagram on the right. This is often termed a deceleration.





EXPLORE 1.2

If the initial velocity is negative, what effect would a positive acceleration have on the car? Would it be moving more quickly or less quickly?

What effect would a negative acceleration have on the car in this situation? Would it be moving more quickly or less quickly?

When the acceleration is constant, the average velocity is simply the average of the initial and final velocities, which is given by the formula $\frac{1}{2}(u + v)$. This can be used to find displacements using the equation for average velocity from Key Point 1.5.

KEY POINT 1.7

If an object has constant acceleration a, initial velocity u and it reaches final velocity v in time t, then the displacement s is given by

$$s = \frac{1}{2}(u + v)t$$

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WORKED EXAMPLE 1.4

A parachutist falls from rest to 49 m s^{-1} over 5 s. Find her acceleration.

Answer

$a = \frac{v - u}{t}$

$= \frac{49 - 0}{5}$

$= 9.8\,\text{m s}^{-2}$ — Make sure you use the correct units, which are m s^{-2}.

TIP

'Rest' means not moving, so velocity is zero.

WORKED EXAMPLE 1.5

A tractor accelerates from 5 m s^{-1} to 9 m s^{-1} at 0.5 m s^{-2}. Find the distance covered by the tractor over this time.

Answer

$a = \frac{v - u}{t}$ — Substitute into $a = \frac{v - u}{t}$ first to find t.

So $0.5 = \frac{9 - 5}{t}$

$0.5t = 4$

$t = 8\,\text{s}$

$s = \frac{1}{2}(u + v)t$ — Substitute into $s = \frac{1}{2}(u + v)t$ to find s.

$= \frac{1}{2}(5 + 9) \times 8$

$= 56\,\text{m}$

EXERCISE 1B

1 A car accelerates from 4 m s^{-1} to 10 m s^{-1} in 3 s at constant acceleration. Find its acceleration.

2 A car accelerates from rest to 10 m s^{-1} in 4 s at constant acceleration. Find its acceleration.

3 A car accelerates from 3 m s^{-1} at an acceleration of 6 m s^{-2}. Find the time taken to reach 12 m s^{-1}.

4 An aeroplane accelerates at a constant rate of 3 m s^{-2} for 5 s from an initial velocity of 4 m s^{-1}. Find its final velocity.

5 A speedboat accelerates at a constant rate of 1.5 m s^{-2} for 4 s, reaching a final velocity of 9 m s^{-1}. Find its initial velocity.

6 A car decelerates at a constant rate of 2 m s^{-2} for 3 s, finishing at a velocity of 8 m s^{-1}. Find its initial velocity.

7 A car accelerates from an initial velocity of 4 m s^{-1} to a final velocity of 8 m s^{-1} at a constant rate of 0.5 m s^{-2}. Find the car's displacement in that time.