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Sue Pemberton , Julianne Hughes , Edited by Julian Gilbey
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Chapter 1
Algebra

In this chapter you will learn how to:

- understand the meaning of $|x|$, sketch the graph of $y = |ax + b|$ and use relations such as $|a| = |b| \Leftrightarrow a^2 = b^2$ and $|x - a| < b \Leftrightarrow a - b < x < a + b$ in the course of solving equations and inequalities
- divide a polynomial, of degree not exceeding 4, by a linear or quadratic polynomial, and identify the quotient and remainder (which may be zero)
- use the factor theorem and the remainder theorem.

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PREREQUISITE KNOWLEDGE

Where it comes from	What you should be able to do	Check your skills
IGCSE® / O Level Mathematics	Perform long division on numbers and find the remainder where necessary.	**1** Using long division, calculate: **a** $4998 \div 14$ **b** $10\,287 \div 27$ **c** $4283 \div 32$
IGCSE / O Level Mathematics	Sketch straight-line graphs.	**2** Sketch the graph of $y = 2x - 5$.

Why do we need to study algebra?

You have previously learnt the general formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, for solving quadratic equations. Versions of this formula were known and used by the Babylonians nearly 4000 years ago.

It took until the 16th century for mathematicians to discover formulae for solving cubic equations, $ax^3 + bx^2 + cx + d = 0$, and quartic equations, $ax^4 + bx^3 + cx^2 + dx + e = 0$. (These formulae are very complicated and there is insufficient space to include them here but you might wish to research them on the internet.)

Mathematicians then spent many years trying to discover a general formula for solving quintic equations, $ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$. Eventually, in 1824, a mathematician managed to prove that no general formula for this exists.

In this chapter you will develop skills for factorising and solving cubic and quartic equations. These types of equations have various applications in the real world. For example, cubic equations are used in thermodynamics and fluid mechanics to model the pressure/volume/temperature behaviour of gases and fluids.

You will also learn about a new type of function, called the modulus function.

WEB LINK

Explore the *Polynomials and rational functions* station on the Underground Mathematics website.

1.1 The modulus function

The **modulus** of a number is the magnitude of the number without a sign attached.

The modulus of 3 is written $|3|$.

$|3| = 3$ and $|-3| = 3$.

It is important to note that the modulus of any number (positive or negative) is always a positive number.

The modulus of a number is also called the **absolute value**.

The modulus of x, written as $|x|$, is defined as:

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

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EXPLORE 1.1

You are given these eight statements:

1. $|a+b| = |a| + |b|$
2. $|a-b| = |a| - |b|$
3. $|ab| = |a| \times |b|$
4. $\left|\frac{a}{b}\right| = |a| \div |b|$, if $b \neq 0$
5. $|a|^2 = a^2$
6. $|a|^n = a^n$, where n is a positive integer
7. $|a+b| \leqslant |a| + |b|$
8. $|a-b| \leqslant |a-c| + |c-b|$

You must decide whether a statement is

Always true | Sometimes true | Never true

- If you think that a statement is either **always true** or **never true**, you must give a clear explanation to justify your answer.
- If you think that a statement is **sometimes true** you must give an example of when it is true and an example of when it is not true.

The statement $|x| = k$, where $k \geqslant 0$, means that $x = k$ or $x = -k$.

This property is used to solve equations that involve modulus functions.

If you are solving equations of the form $|ax+b| = k$, you can solve the equations using

$ax + b = k$ and $ax + b = -k$

If you are solving harder equations of the form $|ax+b| = cx + d$, you can solve the equations using

$ax + b = cx + d$ and $ax + b = -(cx + d)$

When solving these more complicated equations you must always check that your answers satisfy the original equation.

FAST FORWARD

You will see why it is necessary to check your answers when you learn how to sketch graphs of modulus functions in Section 1.2.

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WORKED EXAMPLE 1.1

Solve:

a $|2x-1| = 3$

b $|x-4| = 2x+1$

Answer

a $|2x-1| = 3$

$2x-1 = 3$ or $2x-1 = -3$

$2x = 4$ $\quad$ $2x = -2$

$x = 2$ $\quad$ $x = -1$

CHECK: $|2\times 2-1| = 3$ ✓ and $|2\times(-1)-1| = 3$ ✓

The solution is: $x = -1$ or $x = 2$

b $|x-4| = 2x+1$

$x-4 = 2x+1$ or $x-4 = -(2x+1)$

$x = -5$ $\quad$ $3x = 3$

$x = 1$

CHECK: $|-5-4| = 2\times(-5)+1$ × and $|1-4| = 2\times 1+1$ ✓

The solution is: $x = 1$

EXPLORE 1.2

Using $|a|^2 = a^2$ you can write $a^2 - b^2$ as $|a|^2 - |b|^2$.

Using the difference of two squares you can now write:

$$a^2 - b^2 = (|a| - |b|)(|a| + |b|)$$

Using the previous statement, explain how these three important results can be obtained:

(The symbol ⇔ means 'is equivalent to'.)

- $|a| = |b| \Leftrightarrow a^2 = b^2$
- $|a| > |b| \Leftrightarrow a^2 > b^2$
- $|a| < |b| \Leftrightarrow a^2 < b^2$, if $b \neq 0$.

Worked example 1.2 shows you how to solve equations of the form $|cx+d| = |ex+f|$.

KEY POINT 1.1

To solve equations of the form $|cx+d| = |ex+f|$ we can use the rule:

$|a| = |b| \Leftrightarrow a^2 = b^2$

WORKED EXAMPLE 1.2

Solve the equation $|3x+4| = |x+5|$.

Answer

Method 1

$|3x+4| = |x+5|$

$3x+4 = x+5$ or $3x+4 = -(x+5)$

$2x = 1$ or $4x = -9$

$x = \frac{1}{2}$ $\quad$ $x = -\frac{9}{4}$

CHECK: $\left|3\times\frac{1}{2}+4\right| = \left|\frac{1}{2}+5\right|$ ✓, $\left|3\times\left(-\frac{9}{4}\right)+4\right| = \left|-\frac{9}{4}+5\right|$ ✓

Solution is: $x = \frac{1}{2}$ or $x = -\frac{9}{4}$

Method 2

$|3x+4| = |x+5|$ — Use $|a| = |b| \Leftrightarrow a^2 = b^2$.

$(3x+4)^2 = (x+5)^2$

$9x^2 + 24x + 16 = x^2 + 10x + 25$

$8x^2 + 14x - 9 = 0$ — Factorise.

$(2x-1)(4x+9) = 0$

$x = \frac{1}{2}$ or $x = -\frac{9}{4}$

WORKED EXAMPLE 1.3

Solve $|x+3| + |x+5| = 10$.

Answer

$|x+3| + |x+5| = 10$ — Subtract $|x+5|$ from both sides.

$|x+3| = 10 - |x+5|$ — Split the equation into two parts.

$x+3 = 10 - |x+5|$ ------------ (1)

$x+3 = |x+5| - 10$ ------------ (2)

Using equation (1):

$-7 + x + |x+5| = 0$

$|x+5| = 7 - x$ — Split this equation into two parts.

$x+5 = 7-x$ or $x+5 = -(7-x)$

$2x = 2$ or $0 = -12$ — $0 = -12$ is false.

$x = 1$

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Using equation (2):

$|x+5| = x+13$ — Split this equation into two parts.

$x+5 = x+13$ or $x+5 = -(x+13)$

$0 = 8$ or $2x = -18$ — $0 = 8$ is false.

$x = -9$

The solution is $x = 1$ or $x = -9$.

EXERCISE 1A

1 Solve.

a $|4x-3| = 7$ b $|1-2x| = 5$ c $\left|\frac{3x-2}{5}\right| = 4$

d $\left|\frac{x}{3}+2\right| = 3$ e $\left|\frac{x+2}{3}-\frac{2x}{5}\right| = 2$ f $|2x+7| = 3x$

2 Solve.

a $\left|\frac{2x+1}{x-2}\right| = 5$ b $\left|\frac{3x-1}{x+5}\right| = 1$ c $\left|2-\frac{x+2}{x-3}\right| = 5$

d $|3x-5| = x+1$ e $x+|x+4| = 8$ f $8-|1-2x| = x$

3 Solve.

a $|2x+1| = |x|$ b $|3-2x| = |3x|$ c $|2x-5| = |1-x|$

d $|3x+5| = |1+2x|$ e $|x-5| = 2|x+1|$ f $3|2x-1| = \left|\frac{1}{2}x-3\right|$

4 Solve.

a $|x^2-2| = 7$ b $|5-x^2| = 3-x$ c $|x^2+2x| = x+2$

d $|x^2-3| = 2x+1$ e $|2x^2-5x| = 4-x$ f $|x^2-7x+6| = 6-x$

5 Solve the simultaneous equations.

a $x+2y = 8$
$|x+2|+y = 6$

b $3x+y = 0$
$y = |2x^2-5|$

6 Solve the equation $5|x-1|^2+9|x-1|-2 = 0$.

7 a Solve the equation $x^2-5|x|+6 = 0$.

b Use graphing software to draw the graph of $y = x^2-5|x|+6$.

c Name the equation of the line of symmetry of the curve.

8 Solve the equation $|2x+1|+|2x-1| = 3$.

PS 9 Solve the equation $|3x-2y-11|+2\sqrt{31-8x+5y} = 0$.

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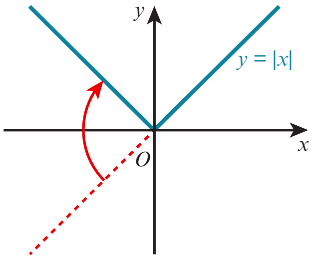
1.2 Graphs of $y = |f(x)|$ where f(x) is linear

Consider drawing the graph of $y = |x|$.

First draw the line $y = x$.

Then reflect, in the x-axis, the part of the line that is below the x-axis.



WORKED EXAMPLE 1.4

Sketch the graph of $y = \left|\frac{1}{2}x - 1\right|$, showing the points where the graph meets the axes.

Use your graph to express $\left|\frac{1}{2}x - 1\right|$ in an alternative form.

Answer

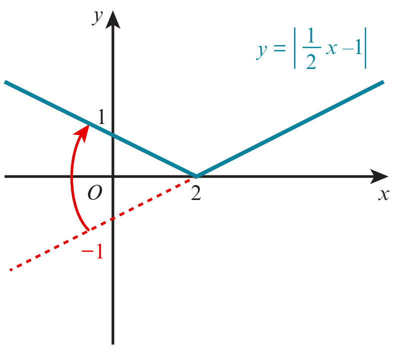
First sketch the graph of $y = \frac{1}{2}x - 1$.

The line has gradient $\frac{1}{2}$ and a y-intercept of -1.

You then reflect in the x-axis the part of the line that is below the x-axis.

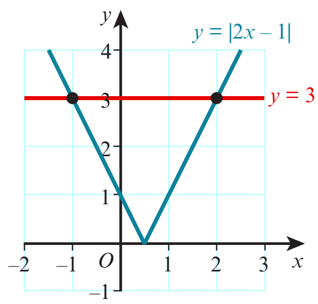
The graph shows that $\left|\frac{1}{2}x - 1\right|$ can be written as:

$$\left|\frac{1}{2}x - 1\right| = \begin{cases} \frac{1}{2}x - 1 & \text{if } x \geqslant 2 \\ -\left(\frac{1}{2}x - 1\right) & \text{if } x < 2 \end{cases}$$



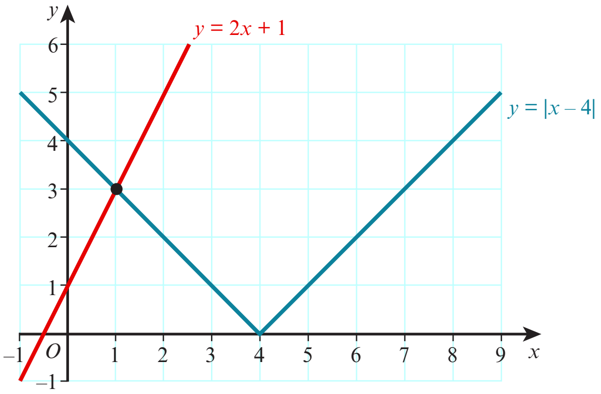
In Worked example 1.1, we found that there were two roots, $x = -1$ and $x = 2$, to the equation $|2x - 1| = 3$.

These can also be found graphically by finding the x-coordinates of the points of intersection of the graphs of $y = |2x - 1|$ and $y = 3$ as shown.



Also in Worked example 1.1, we found that there was only one root, $x = 1$, to the equation $|x - 4| = 2x + 1$.

This root can be found graphically by finding the x-coordinates of the points of intersection of the graphs of $y = |x - 4|$ and $y = 2x + 1$ as shown.



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EXERCISE 1B

1 Sketch the graphs of each of the following functions showing the coordinates of the points where the graph meets the axes. Express each function in an alternative form.

a $y = |x + 2|$ **b** $y = |3 - x|$ **c** $y = \left|5 - \frac{1}{2}x\right|$

2 a Complete the table of values for $y = |x - 3| + 2$.

x	0	1	2	3	4	5	6
y	5		3				

b Draw the graph of $y = |x - 3| + 2$ for $0 \leqslant x \leqslant 6$.

c Describe the transformation that maps the graph of $y = |x|$ onto the graph of $y = |x - 3| + 2$.

3 Describe fully the transformation (or combination of transformations) that maps the graph of $y = |x|$ onto each of these functions.

a $y = |x + 1| + 2$ **b** $y = |x - 5| - 2$ **c** $y = 2 - |x|$

d $y = |2x| - 3$ **e** $y = 1 - |x + 2|$ **f** $y = 5 - 2|x|$

4 Sketch the graphs of each of the functions in question 3. For each graph, state the coordinates of the vertex.

5 $f(x) = |5 - 2x| + 3$ for $2 \leqslant x \leqslant 8$

Find the range of function f.

6 a Sketch the graph of $y = 2|x - 2| + 1$ for $-2 < x < 6$, showing the coordinates of the vertex and the y-intercept.

b On the same diagram, sketch the graph of $y = x + 2$.

c Use your graph to solve the equation $2|x - 2| + 1 = x + 2$.

7 a Sketch the graph of $y = |x - 2|$ for $-3 < x < 6$, showing the coordinates of the vertex and the y-intercept.

b On the same diagram, sketch the graph of $y = |1 - 2x|$.

c Use your graph to solve the equation $|x - 2| = |1 - 2x|$.

8 a Sketch the graph of $y = |x + 1| + |x - 1|$.

b Use your graph to solve the equation $|x + 1| + |x - 1| = 4$.

1.3 Solving modulus inequalities

Two useful properties that can be used when solving modulus inequalities are:

$$|a| \leqslant b \Leftrightarrow -b \leqslant a \leqslant b \quad \text{and} \quad |a| \geqslant b \Leftrightarrow a \leqslant -b \ \text{ or } \ a \geqslant b$$

The following examples illustrate the different methods that can be used when solving modulus inequalities.

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WORKED EXAMPLE 1.5

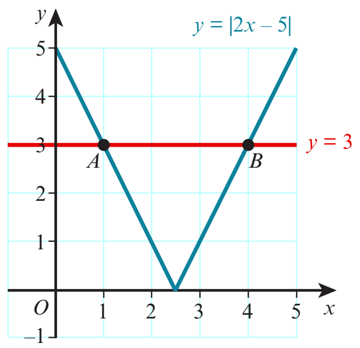
Solve $|2x - 5| < 3$.

Answer

Method 1: Using algebra

$$|2x - 5| < 3$$
$$-3 < 2x - 5 < 3$$
$$2 < 2x < 8$$
$$1 < x < 4$$

Method 2: Using a graph



The graphs of $y = |2x - 5|$ and $y = 3$ intersect at the points A and B.

$$|2x - 5| = \begin{cases} 2x - 5 & \text{if } x \geqslant 2\frac{1}{2} \\ -(2x - 5) & \text{if } x < 2\frac{1}{2} \end{cases}$$

At A, the line $y = -(2x - 5)$ intersects the line $y = 3$.

$$-(2x - 5) = 3$$
$$-2x + 5 = 3$$
$$2x = 2$$
$$x = 1$$

At B, the line $y = 2x - 5$ intersects the line $y = 3$.

$$2x - 5 = 3$$
$$2x = 8$$
$$x = 4$$

To solve the inequality $|2x - 5| < 3$ you must find where the graph of the function $y = |2x - 5|$ is below the graph of $y = 3$.

Hence, $1 < x < 4$.

WORKED EXAMPLE 1.6

Solve the inequality $|2x - 1| \geqslant |3 - x|$.

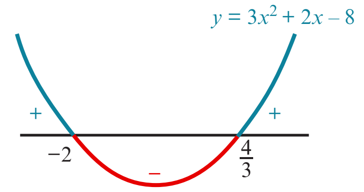
Answer

Method 1: Using algebra

$$|2x - 1| \geqslant |3 - x|$$ Use $|a| \geqslant |b| \Leftrightarrow a^2 \geqslant b^2$.
$$(2x - 1)^2 \geqslant (3 - x)^2$$
$$4x^2 - 4x + 1 \geqslant 9 - 6x + x^2$$
$$3x^2 + 2x - 8 \geqslant 0$$ Factorise.
$$(3x - 4)(x + 2) \geqslant 0$$

Critical values are -2 and $\frac{4}{3}$.

Hence, $x \leqslant -2$ or $x \geqslant \frac{4}{3}$.



Method 2: Using a graph

The graphs of $y = |2x-1|$ and $y = |3-x|$ intersect at the points A and B.

$$|2x-1| = \begin{cases} 2x-1 & \text{if } x \geqslant \frac{1}{2} \\ -(2x-1) & \text{if } x < \frac{1}{2} \end{cases}$$

$$|3-x| = |x-3| = \begin{cases} x-3 & \text{if } x \geqslant 3 \\ -(x-3) & \text{if } x < 3 \end{cases}$$

At A, the line $y = -(2x-1)$ intersects the line $y = -(x-3)$.

$2x - 1 = x - 3$

$x = -2$

At B, the line $y = 2x - 1$ intersects the line $y = -(x-3)$.

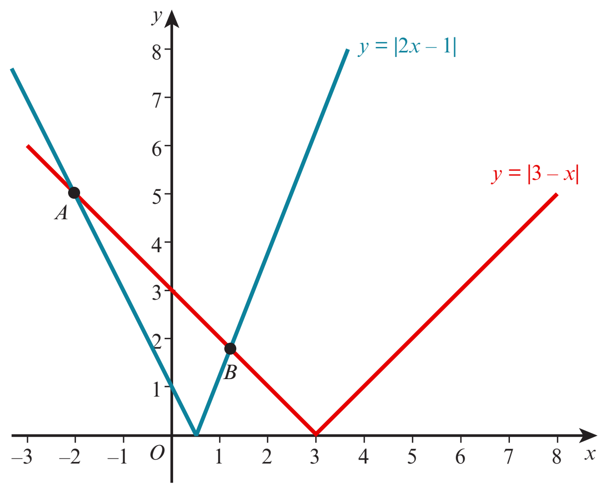
$2x - 1 = -(x-3)$

$3x = 4$

$x = \frac{4}{3}$

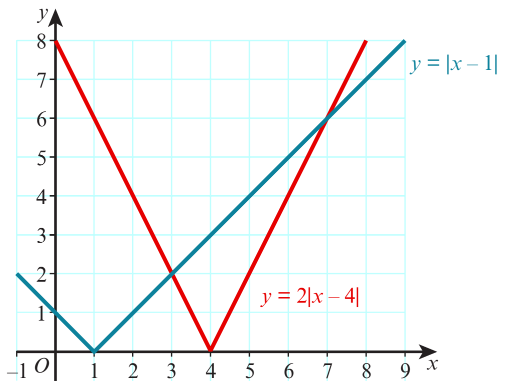
To solve the inequality $|2x-1| \geqslant |3-x|$ you must find where the graph of the function $y = |2x-1|$ is above the graph of $y = |3-x|$.

Hence, $x \leqslant -2$ or $x \geqslant \frac{4}{3}$.



EXERCISE 1C

1



The graphs of $y = |x-1|$ and $y = 2|x-4|$ are shown on the grid.

Write down the solution to the inequality $|x-1| > 2|x-4|$.

2 **a** On the same axes, sketch the graphs of $y = |2x-1|$ and $y = 4 - |x-1|$.

b Solve the inequality $|2x-1| > 4 - |x-1|$.

