Chapter 1
Quadratics

In this chapter you will learn how to:

- carry out the process of completing the square for a quadratic polynomial $ax^2 + bx + c$ and use a completed square form
- find the discriminant of a quadratic polynomial $ax^2 + bx + c$ and use the discriminant
- solve quadratic equations, and quadratic inequalities, in one unknown
- solve by substitution a pair of simultaneous equations of which one is linear and one is quadratic
- recognise and solve equations in x that are quadratic in some function of x
- understand the relationship between a graph of a quadratic function and its associated algebraic equation, and use the relationship between points of intersection of graphs and solutions of equations.

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PREREQUISITE KNOWLEDGE

Where it comes from	What you should be able to do	Check your skills
IGCSE® / O Level Mathematics	Solve quadratic equations by factorising.	**1** Solve: **a** $x^2 + x - 12 = 0$ **b** $x^2 - 6x + 9 = 0$ **c** $3x^2 - 17x - 6 = 0$
IGCSE / O Level Mathematics	Solve linear inequalities.	**2** Solve: **a** $5x - 8 > 2$ **b** $3 - 2x \leqslant 7$
IGCSE / O Level Mathematics	Solve simultaneous linear equations.	**3** Solve: **a** $2x + 3y = 13$ $7x - 5y = -1$ **b** $2x - 7y = 31$ $3x + 5y = -31$
IGCSE / O Level Additional Mathematics	Carry out simple manipulation of surds.	**4** Simplify: **a** $\sqrt{20}$ **b** $(\sqrt{5})^2$ **c** $\frac{8}{\sqrt{2}}$

Why do we study quadratics?

At IGCSE / O Level, you will have learnt about straight-line graphs and their properties. They arise in the world around you. For example, a cell phone contract might involve a fixed monthly charge and then a certain cost per minute for calls: the monthly cost, y, is then given as $y = mx + c$, where c is the fixed monthly charge, m is the cost per minute and x is the number of minutes used.

Quadratic functions are of the form $y = ax^2 + bx + c$ (where $a \neq 0$) and they have interesting properties that make them behave very differently from linear functions. A quadratic function has a maximum or a minimum value, and its graph has interesting symmetry. Studying quadratics offers a route into thinking about more complicated functions such as $y = 7x^5 - 4x^4 + x^2 + x + 3$.

You will have plotted graphs of quadratics such as $y = 10 - x^2$ before starting your A Level course. These are most familiar as the shape of the path of a ball as it travels through the air (called its *trajectory*). Discovering that the trajectory is a quadratic was one of Galileo's major successes in the early 17th century. He also discovered that the vertical motion of a ball thrown straight upwards can be modelled by a quadratic, as you will learn if you go on to study the Mechanics component.

WEB LINK

Try the *Quadratics* resource on the Underground Mathematics website (www.underground mathematics.org).

 www.cambridge.org

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1.1 Solving quadratic equations by factorisation

You already know the factorisation method and the quadratic formula method to solve quadratic equations algebraically.

This section consolidates and builds on your previous work on solving quadratic equations by factorisation.

EXPLORE 1.1

$2x^2 + 3x - 5 = (x - 1)(x - 2)$

This is Rosa's solution to the previous equation:

Factorise the left-hand side: $(x-1)(2x+5) = (x-1)(x-2)$

Divide both sides by $(x-1)$: $2x+5 = x-2$

Rearrange: $x = -7$

Discuss her solution with your classmates and explain why her solution is not fully correct.

Now solve the equation correctly.

WORKED EXAMPLE 1.1

Solve:

a $6x^2 + 5 = 17x$

b $9x^2 - 39x - 30 = 0$

Answer

a

$6x^2 + 5 = 17x$ — Write in the form $ax^2 + bx + c = 0$.

$6x^2 - 17x + 5 = 0$ — Factorise.

$(2x - 5)(3x - 1) = 0$ — Use the fact that if $pq = 0$, then $p = 0$ or $q = 0$.

$2x - 5 = 0$ or $3x - 1 = 0$ — Solve.

$x = \frac{5}{2}$ or $x = \frac{1}{3}$

b

$9x^2 - 39x - 30 = 0$ — Divide both sides by the common factor of 3.

$3x^2 - 13x - 10 = 0$ — Factorise.

$(3x + 2)(x - 5) = 0$

$3x + 2 = 0$ or $x - 5 = 0$ — Solve.

$x = -\frac{2}{3}$ or $x = 5$

TIP

Divide by a common factor first, if possible.

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WORKED EXAMPLE 1.2

Solve $\frac{21}{2x} - \frac{2}{x+3} = 1$.

Answer

$\frac{21}{2x} - \frac{2}{x+3} = 1$ Multiply both sides by $2x(x + 3)$.

$21(x + 3) - 4x = 2x(x + 3)$ Expand brackets and rearrange.

$2x^2 - 11x - 63 = 0$ Factorise.

$(2x + 7)(x - 9) = 0$

$2x + 7 = 0$ or $x - 9 = 0$ Solve.

$x = -\frac{7}{2}$ or $x = 9$

WORKED EXAMPLE 1.3

Solve $\frac{3x^2 + 26x + 35}{x^2 + 8} = 0$.

Answer

$\frac{3x^2 + 26x + 35}{x^2 + 8} = 0$ Multiply both sides by $x^2 + 8$.

$3x^2 + 26x + 35 = 0$ Factorise.

$(3x + 5)(x + 7) = 0$

$3x + 5 = 0$ or $x + 7 = 0$ Solve.

$x = -\frac{5}{3}$ or $x = -7$

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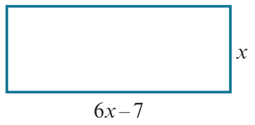
WORKED EXAMPLE 1.4

A rectangle has sides of length x cm and $(6x - 7)$ cm.

The area of the rectangle is 90 cm^2.

Find the lengths of the sides of the rectangle.



Answer

Area $= x(6x - 7) = 6x^2 - 7x = 90$ ········ Rearrange.

$6x^2 - 7x - 90 = 0$ ········ Factorise.

$(2x - 9)(3x + 10) = 0$

$2x - 9 = 0$ or $3x + 10 = 0$ ········ Solve.

$x = \frac{9}{2}$ or $x = -\frac{10}{3}$ ········ Length is a positive quantity, so $x = 4\frac{1}{2}$.

When $x = 4\frac{1}{2}$, $6x - 7 = 20$.

The rectangle has sides of length $4\frac{1}{2}$ cm and 20 cm.

EXPLORE 1.2

A $4^{(2x^2 + x - 6)} = 1$

B $(x^2 - 3x + 1)^6 = 1$

C $(x^2 - 3x + 1)^{(2x^2 + x - 6)} = 1$

1 Discuss with your classmates how you would solve each of these equations.

2 Solve:

a equation **A** **b** equation **B** **c** equation **C**

3 State how many values of x satisfy:

a equation **A** **b** equation **B** **c** equation **C**

4 Discuss your results.

TIP

Remember to check each of your answers.

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EXERCISE 1A

1 Solve by factorisation.

a $x^2 + 3x - 10 = 0$ **b** $x^2 - 7x + 12 = 0$ **c** $x^2 - 6x - 16 = 0$

d $5x^2 + 19x + 12 = 0$ **e** $20 - 7x = 6x^2$ **f** $x(10x - 13) = 3$

2 Solve:

a $x - \dfrac{6}{x-5} = 0$ **b** $\dfrac{2}{x} + \dfrac{3}{x+2} = 1$

c $\dfrac{5x+1}{4} - \dfrac{2x-1}{2} = x^2$ **d** $\dfrac{5}{x+3} + \dfrac{3x}{x+4} = 2$

e $\dfrac{3}{x+1} + \dfrac{1}{x(x+1)} = 2$ **f** $\dfrac{3}{x+2} + \dfrac{1}{x-1} = \dfrac{1}{(x+1)(x+2)}$

3 Solve:

a $\dfrac{3x^2 + x - 10}{x^2 - 7x + 6} = 0$ **b** $\dfrac{x^2 + x - 6}{x^2 + 5} = 0$ **c** $\dfrac{x^2 - 9}{7x + 10} = 0$

d $\dfrac{x^2 - 2x - 8}{x^2 + 7x + 10} = 0$ **e** $\dfrac{6x^2 + x - 2}{x^2 + 7x + 4} = 0$ **f** $\dfrac{2x^2 + 9x - 5}{x^4 + 1} = 0$

4 Find the real solutions of the following equations.

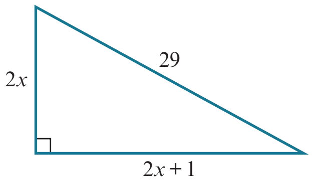
a $8^{(x^2 + 2x - 15)} = 1$ **b** $4^{(2x^2 - 11x + 15)} = 1$ **c** $2^{(x^2 - 4x + 6)} = 8$

d $3^{(2x^2 + 9x + 2)} = \dfrac{1}{9}$ **e** $(x^2 + 2x - 14)^5 = 1$ **f** $(x^2 - 7x + 11)^8 = 1$

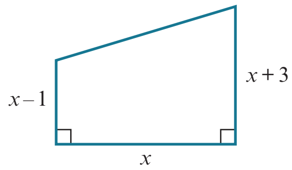
5 The diagram shows a right-angled triangle with sides $2x$ cm, $(2x + 1)$ cm and 29 cm.

a Show that $2x^2 + x - 210 = 0$.

b Find the lengths of the sides of the triangle.



6 The area of the trapezium is $35.75\,\text{cm}^2$.
Find the value of x.



PS **7** Solve $(x^2 - 11x + 29)^{(6x^2 + x - 2)} = 1$.

TIP

Check that your answers satisfy the original equation.

WEB LINK

Try the *Factorisable quadratics* resource on the Underground Mathematics website.

1.2 Completing the square

Another method we can use for solving quadratic equations is **completing the square**.

The method of completing the square aims to rewrite a quadratic expression using only one occurrence of the variable, making it an easier expression to work with.

If we expand the expressions $(x + d)^2$ and $(x - d)^2$, we obtain the results:

$$(x + d)^2 = x^2 + 2dx + d^2 \quad \text{and} \quad (x - d)^2 = x^2 - 2dx + d^2$$

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Rearranging these gives the following important results:

KEY POINT 1.1

$x^2 + 2dx = (x + d)^2 - d^2$ and $x^2 - 2dx = (x - d)^2 - d^2$

To complete the square for $x^2 + 10x$, we can use the first of the previous results as follows:

$$10 \div 2 = 5$$
$$x^2 + 10x = (x + 5)^2 - 5^2$$
$$x^2 + 10x = (x + 5)^2 - 25$$

To complete the square for $x^2 + 8x - 7$, we again use the first result applied to the $x^2 + 8x$ part, as follows:

$$8 \div 2 = 4$$
$$x^2 + 8x - 7 = (x + 4)^2 - 4^2 - 7$$
$$x^2 + 8x - 7 = (x + 4)^2 - 23$$

To complete the square for $2x^2 - 12x + 5$, we must first take a factor of 2 out of the first two terms, so:

$$2x^2 - 12x + 5 = 2(x^2 - 6x) + 5$$
$$6 \div 2 = 3$$
$$x^2 - 6x = (x - 3)^2 - 3^2, \text{ giving}$$
$$2x^2 - 12x + 5 = 2[(x - 3)^2 - 9] + 5 = 2(x - 3)^2 - 13$$

We can also use an algebraic method for completing the square, as shown in Worked example 1.5.

WORKED EXAMPLE 1.5

Express $2x^2 - 12x + 3$ in the form $p(x - q)^2 + r$, where p, q and r are constants to be found.

Answer

$$2x^2 - 12x + 3 = p(x - q)^2 + r$$

Expanding the brackets and simplifying gives:

$$2x^2 - 12x + 3 = px^2 - 2pqx + pq^2 + r$$

Comparing coefficients of x^2, coefficients of x and the constant gives

$2 = p$ --------- (1) $\quad -12 = -2pq$ --------- (2) $\quad 3 = pq^2 + r$ ------ (3)

Substituting $p = 2$ in equation (2) gives $q = 3$

Substituting $p = 2$ and $q = 3$ in equation (3) therefore gives $r = -15$

$$2x^2 - 12x + 3 = 2(x - 3)^2 - 15$$

WORKED EXAMPLE 1.6

Express $4x^2 + 20x + 5$ in the form $(ax + b)^2 + c$, where a, b and c are constants to be found.

Answer

$4x^2 + 20x + 5 = (ax + b)^2 + c$

Expanding the brackets and simplifying gives:

$4x^2 + 20x + 5 = a^2x^2 + 2abx + b^2 + c$

Comparing coefficients of x^2, coefficients of x and the constant gives

$4 = a^2$ ------(1) $20 = 2ab$ ------ (2) $5 = b^2 + c$ ------ (3)

Equation (1) gives $a = \pm 2$.

Substituting $a = 2$ into equation (2) gives $b = 5$.

Substituting $b = 5$ into equation (3) gives $c = -20$.

$4x^2 + 20x + 5 = (2x + 5)^2 - 20$

Alternatively:

Substituting $a = -2$ into equation (2) gives $b = -5$.

Substituting $b = -5$ into equation (3) gives $c = -20$.

$4x^2 + 20x + 5 = (-2x - 5)^2 - 20 = (2x + 5)^2 - 20$

WORKED EXAMPLE 1.7

Use completing the square to solve the equation $\dfrac{5}{x+2} + \dfrac{3}{x-5} = 1$.
Leave your answers in surd form.

Answer

$\dfrac{5}{x+2} + \dfrac{3}{x-5} = 1$ — Multiply both sides by $(x + 2)(x - 5)$.

$5(x - 5) + 3(x + 2) = (x + 2)(x - 5)$ — Expand brackets and collect terms.

$x^2 - 11x + 9 = 0$ — Complete the square.

$$\left(x - \frac{11}{2}\right)^2 - \left(\frac{11}{2}\right)^2 + 9 = 0$$

$$\left(x - \frac{11}{2}\right)^2 = \frac{85}{4}$$

$$x - \frac{11}{2} = \pm\sqrt{\frac{85}{4}}$$

$$x = \frac{11}{2} \pm \frac{\sqrt{85}}{2}$$

$$x = \frac{1}{2}(11 \pm \sqrt{85})$$

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EXERCISE 1B

1 Express each of the following in the form $(x+a)^2+b$.

a x^2-6x b x^2+8x c x^2-3x d x^2+15x

e x^2+4x+8 f x^2-4x-8 g x^2+7x+1 h x^2-3x+4

2 Express each of the following in the form $a(x+b)^2+c$.

a $2x^2-12x+19$ b $3x^2-12x-1$ c $2x^2+5x-1$ d $2x^2+7x+5$

3 Express each of the following in the form $a-(x+b)^2$.

a $4x-x^2$ b $8x-x^2$ c $4-3x-x^2$ d $9+5x-x^2$

4 Express each of the following in the form $p-q(x+r)^2$.

a $7-8x-2x^2$ b $3-12x-2x^2$ c $13+4x-2x^2$ d $2+5x-3x^2$

5 Express each of the following in the form $(ax+b)^2+c$.

a $9x^2-6x-3$ b $4x^2+20x+30$ c $25x^2+40x-4$ d $9x^2-42x+61$

6 Solve by completing the square.

a $x^2+8x-9=0$ b $x^2+4x-12=0$ c $x^2-2x-35=0$

d $x^2-9x+14=0$ e $x^2+3x-18=0$ f $x^2+9x-10=0$

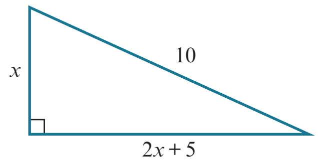
7 Solve by completing the square. Leave your answers in surd form.

a $x^2+4x-7=0$ b $x^2-10x+2=0$ c $x^2+8x-1=0$

d $2x^2-4x-5=0$ e $2x^2+6x+3=0$ f $2x^2-8x-3=0$

8 Solve $\dfrac{5}{x+2}+\dfrac{3}{x-4}=2$. Leave your answers in surd form.

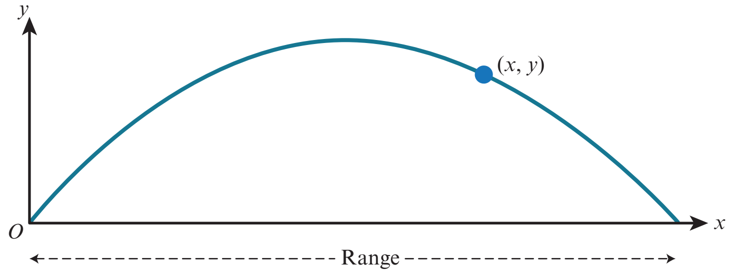
PS 9 The diagram shows a right-angled triangle with sides x m, $(2x+5)$ m and 10 m.



Find the value of x. Leave your answer in surd form.

PS 10 Find the real solutions of the equation $(3x^2+5x-7)^4=1$.

PS 11 The path of a projectile is given by the equation $y=(\sqrt{3})x-\dfrac{49x^2}{9000}$, where x and y are measured in metres.



a Find the range of this projectile.

b Find the maximum height reached by this projectile.

TIP

You will learn how to derive formulae such as this if you go on to study Further Mathematics.

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1.3 The quadratic formula

We can solve quadratic equations using the **quadratic formula**.

If $ax^2 + bx + c = 0$, where a, b and c are constants and $a \neq 0$, then

KEY POINT 1.2

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The quadratic formula can be proved by completing the square for the equation $ax^2 + bx + c = 0$:

$$ax^2 + bx + c = 0 \qquad \text{Divide both sides by } a.$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad \text{Complete the square.}$$

$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0 \qquad \text{Rearrange the equation.}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \qquad \text{Write the right-hand side as a single fraction.}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \qquad \text{Find the square root of both sides.}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \qquad \text{Subtract } \frac{b}{2a} \text{ from both sides.}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \qquad \text{Write the right-hand side as a single fraction.}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

WORKED EXAMPLE 1.8

Solve the equation $6x^2 - 3x - 2 = 0$.

Write your answers correct to 3 significant figures.

Answer

Using $a = 6$, $b = -3$ and $c = -2$ in the quadratic formula gives:

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \times 6 \times (-2)}}{2 \times 6}$$

$$x = \frac{3 + \sqrt{57}}{12} \text{ or } x = \frac{3 - \sqrt{57}}{12}$$

$x = 0.879$ or $x = -0.379$ (to 3 significant figures)

