Part One

Theory

1
Analytical Methods

1.1 Setting and basic terminology

We will deal with maps

$$x \mapsto f(x), \quad x \in \mathbb{R}^n, \tag{1.1}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is sufficiently smooth, i.e., has all required continuous partial derivatives with respect to its arguments.[1] To simplify our presentation, we assume that f is a diffeomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^n$, so that its inverse $f^{-1}\colon \mathbb{R}^n \to \mathbb{R}^n$ is globally defined and smooth. A sequence of points $x_n \in \mathbb{R}^n$ is called an *orbit* of (1.1) if

$$x_{k+1} = f(x_k), \qquad k \in \mathbb{Z}.$$

One says that $x_0 \in \mathbb{R}^n$ is a starting point of the orbit. In general, an orbit can be finite, i.e., undefined starting from some (positive or negative) k. The part of an orbit with $k \geq 0$ is called the *forward orbit*. If f is invertible, the *backward orbit* is uniquely defined.

A *fixed point* x_0 satisfies $f(x_0) = x_0$. The orbit starting at a fixed point x_0 is constant:

$$\ldots, x_0, x_0, x_0, \ldots \; .$$

A nonconstant K-periodic orbit $\{x_k\}$, i.e., such that

$$x_K = x_0,$$

where $K > 1$ is the minimal integer possible, is called a *cycle* with *period* K or *K-periodic orbit*. A cycle with period K defines a set of K distinct points,

$$C = \left\{x_0, f(x_0), f^{(2)}(x_0), \ldots, f^{(K-1)}(x_0)\right\},$$

[1] If f is only defined on an open region $U \subset \mathbb{R}^n$ and one is interested in studying dynamics generated by (1.1), then, usually, it is possible to extend f to the whole state space and study a smooth map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ and restrict to U.

with $x_0 = f^{(K)}(x_0)$. Here, $f^{(k)}$ denotes the composition of k copies of f, also called the kth *iterate* of f. Each point in C is a fixed point of $f^{(K)}$.

A subset $S \subset \mathbb{R}^n$ is said to be *invariant* if any orbit starting at $x_0 \in S$ is located in S, i.e., $f^{(k)}(x_0) \in S$ for all $k \in \mathbb{Z}$. Fixed points and cycles are the simplest invariant sets, but more complicated ones exist, e.g., *invariant manifolds* (closed curves, tori) and *fractal invariant sets*.

Let S be an invariant set of a diffeomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^n$. The set

$$W^s(S) := \{x \in \mathbb{R}^n : f^{(k)}(x) \to S \ \text{ as } \ k \to \infty\}$$

is called the *stable set* of S. It is composed of all points converging to S under iteration of f. Similarly,

$$W^u(S) := \{x \in \mathbb{R}^n : f^{(-k)}(x) \to S \ \text{ as } \ k \to \infty\}$$

is called the *unstable set* of S.

A fixed point x_0 of (1.1) is called *hyperbolic* if the Jacobian matrix $A = f_x(x_0) := Df(x_0)$ is nonsingular and has no eigenvalues with $|\lambda| = 1$. If x_0 is hyperbolic, A has n_s *stable eigenvalues* with $|\lambda| < 1$ and n_u *unstable eigenvalues* with $|\lambda| > 1$ with $n_s + n_u = n$. Denote by E^s (E^u) the generalized invariant eigenspace of A corresponding to the union of its stable (unstable) eigenvalues.

Theorem 1.1 (Local Stable and Unstable Invariant Manifolds (Palis and de Melo, 1982)) *Near a hyperbolic fixed point x_0, the map* (1.1) *has two smooth embedded invariant manifolds $W^s(x_0)$ and $W^u(x_0)$ that are tangent at x_0 to the eigenspaces E^s and E^u, respectively.*

The next key notion is that of the *equivalence* of maps. We introduce another map

$$x \mapsto g(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

where $g\colon \mathbb{R}^n \to \mathbb{R}^n$ is sufficiently smooth. The maps (1.1) and (1.2) are *topologically equivalent* if there is a homeomorphism $h\colon \mathbb{R}^n \to \mathbb{R}^n$ that maps orbits of (1.1) onto orbits of (1.2). Analytically, this means that

$$f(x) = h^{-1}(g(h(x)), \quad x \in \mathbb{R}^n,$$

or, equivalently, but easier in practice,

$$h(f(x)) = g(h(x)), \quad x \in \mathbb{R}^n.$$

The number and stability of invariant sets are the same for both maps. If the homeomorphism h is a diffeomorphism, we call the two maps *smoothly equivalent*. One can consider two smoothly equivalent maps as one map written in

Cambridge University Press
978-1-108-49967-5 — Numerical Bifurcation Analysis of Maps
Yuri A. Kuznetsov , Hil G. E. Meijer
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two different coordinate systems. If we restrict our attention to an open neighborhood U of a fixed point or a cycle, we say that the corresponding equivalence is *local*.

Theorem 1.2 (Grobman–Hartman) *Consider a smooth map*

$$x \mapsto Ax + F(x), \quad x \in \mathbb{R}^n, \tag{1.3}$$

where A is an $n \times n$ matrix and $F(x) = O(\|x\|^2)$. If $x = 0$ is a hyperbolic fixed point of (1.3)*, then* (1.3) *is locally topologically equivalent near this point to its linearization*

$$x \mapsto Ax, \quad x \in \mathbb{R}^n.$$

Consider now a family of maps

$$x \mapsto f(x, \alpha), \quad x \in \mathbb{R}^n, \ \alpha \in \mathbb{R}^p, \tag{1.4}$$

where $f\colon \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is smooth. The parameter point $\alpha_0 \in \mathbb{R}^p$ is called a *bifurcation point* if arbitrarily close to it there is $\alpha \in \mathbb{R}^p$ such that (1.4) is not topologically equivalent to

$$x \mapsto f(x, \alpha_0), \quad x \in \mathbb{R}^n,$$

in some domain $U \subset \mathbb{R}^n$. The appearance of a topologically nonequivalent map under a variation of parameters is called a *bifurcation*. Our main goal in this book is to classify and study local bifurcations occurring in generic one- and two-parameter families of smooth maps, and to provide the necessary analytical and numerical tools to analyze these bifurcations in concrete maps. Here, "local" means happening in a small but fixed neighborhood of a fixed point. The minimal number of parameters required to meet a particular bifurcation in a generic family (1.4) is called the *codimension* of the bifurcation. Hence, we focus on a systematic study of local codim 1 and 2 bifurcations. It must be noted immediately that global bifurcations of codim 1 involving cycles and more complicated invariant sets may occur near local codim 2 bifurcation points. We treat the most important aspects of these global bifurcations.

It should also be clear that hyperbolic fixed points do not bifurcate. Indeed, in a smooth family (1.4), a hyperbolic fixed point can only move slightly under small parameter variations, and the local orbit structure near this point remains unchanged due to the Grobman–Hartman Theorem 1.2. Thus, only non-hyperbolic fixed points require further analysis.

Cambridge University Press
978-1-108-49967-5 — Numerical Bifurcation Analysis of Maps
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1.2 Center manifold reduction

Consider a smooth map

$$x \mapsto Ax + F(x), \quad x \in \mathbb{R}^n, \tag{1.5}$$

where A is a nonsingular $n \times n$ matrix and $F(x) = O(\|x\|^2)$. This map has a fixed point $x = 0$ and we would like to study the orbit structure near the origin. Now, suppose that $x = 0$ is a nonhyperbolic fixed point, so that there are in general $n_c > 0$ *critical* eigenvalues of A satisfying $|\lambda| = 1$, n_s *stable* eigenvalues with $|\lambda| < 1$, and n_u *unstable* eigenvalues with $|\lambda| > 1$. Counting these eigenvalues with their algebraic multiplicities, we have $n_c + n_s + n_u = n$. Let E^c, E^s and E^u be the generalized invariant eigenspaces of A corresponding to the critical, stable, and unstable eigenvalues. The following direct-sum decomposition holds: $\mathbb{R}^n = E^c \oplus E^s \oplus E^u$.

It turns out that the map (1.5) possesses an invariant manifold near $x = 0$.

Theorem 1.3 (Center Manifold) *There exists an invariant manifold W_0^c locally defined near $x = 0$ for* (1.5) *with* $\dim W_0^c = n_c$ *that is tangent to E^c at $x = 0$ and has the same (finite) smoothness as F.*

The manifold W_0^c is called the *center manifold*. In general, it is not unique. The map (1.5) is smoothly (linearly) equivalent to the map

$$\begin{pmatrix} \xi \\ u \\ v \end{pmatrix} \mapsto \begin{pmatrix} A_0\xi + F_0(\xi, u, v) \\ A_1 u + F_1(\xi, u, v) \\ A_2 v + F_2(\xi, u, v) \end{pmatrix}, \tag{1.6}$$

where the components of $\xi \in \mathbb{R}^{n_c}$ are coordinates in E^c, the components of $u \in \mathbb{R}^{n_s}$ are coordinates in E^s, and the components of $v \in \mathbb{R}^{n_u}$ are coordinates in E^u. According to Theorem 1.3, the center manifold W_0^c can be represented locally by a graph of a smooth mapping

$$H\colon \mathbb{R}^{n_c} \to \mathbb{R}^{n_s} \times \mathbb{R}^{n_u}, \quad H(0) = 0, H_\xi(0) := DH(0) = 0$$

(see Figure 1.1). In this setting, we have the following theorem.

Theorem 1.4 (Reduction Principle) *The map* (1.6) *is locally topologically equivalent near the origin to*

$$\begin{pmatrix} \xi \\ u \\ v \end{pmatrix} \mapsto \begin{pmatrix} A_0\xi + F_0(\xi, H(\xi)) \\ A_1 u \\ A_2 v \end{pmatrix}. \tag{1.7}$$

This theorem states that dynamics along the stable and unstable subspaces are separated and are determined by the linear maps $u \mapsto A_1 u$ and $v \mapsto A_2 v$,

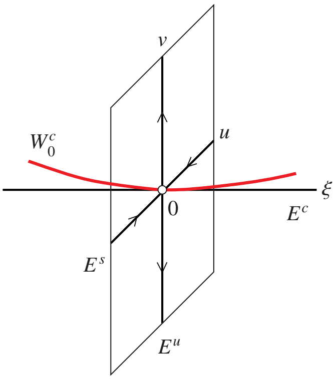

Figure 1.1 Critical center manifold W_0^c for $n_c = n_s = n_u = 1$.

so that the center manifold is *normally hyperbolic*. These dynamics are trivial since all eigenvalues of A_1 satisfy $|\lambda| < 1$, while for those of A_2 we have $|\lambda| > 1$. The dynamics on the center manifold is governed by the nonlinear n_c-dimensional map $\xi \mapsto A_0\xi + f_0(\xi, H(\xi))$, where the linear part has all its n_c eigenvalues on the unit circle. This map is called the *restriction* of (1.6) to its center manifold W_0^c. While the center manifold may not be unique, all such manifolds are represented by functions H having coinciding Taylor expansions. This leads to restricted equations, which can only differ by "flat" functions.

Thus, the analysis of the map (1.5) reduces to that of its restriction to the center manifold. Since the number of critical eigenvalues is usually small, we achieve a considerable simplification.

For a smooth family of smooth maps

$$x \mapsto f(x,\alpha), \quad x \in \mathbb{R}^n,\ \alpha \in \mathbb{R}^p, \tag{1.8}$$

where $f(x,0) = Ax + F(x)$ as in (1.5), there exists a smooth continuation of W_0^c for small $|\alpha|$, i.e., a family of locally defined invariant normally hyperbolic manifolds $W_\alpha^c \subset \mathbb{R}^n$, carrying all interesting local dynamics of $x \mapsto f(x,\alpha)$. This can be shown by considering the *extended* map

$$\begin{pmatrix} x \\ \alpha \end{pmatrix} \mapsto \begin{pmatrix} f(x,\alpha) \\ \alpha \end{pmatrix}, \quad (x,\alpha) \in \mathbb{R}^n \times \mathbb{R}^p, \tag{1.9}$$

and applying Theorem 1.3 to this map. Indeed, for this map, the point $(x,\alpha) = (0,0)$ is nonhyperbolic with $n_c + p$ eigenvalues on the unit circle. It has therefore a $(n_c + p)$-dimensional center manifold with n_c-dimensional α-slices defining W_α^c.

1.3 Normal forms

A smooth map near a fixed point, e.g., the restriction of some map to a center manifold, can be simplified by nonlinear transformations. There is a systematic method to remove as many terms as possible from the Taylor expansion of the map. This method is called *Poincaré normalization*.

Let H_k be the linear space of vector-valued functions whose components are homogeneous polynomials of order k. Consider a smooth map

$$x \mapsto Ax + f^{(2)}(x) + f^{(3)}(x) + \cdots, \quad x \in \mathbb{R}^n, \tag{1.10}$$

where $f^{(k)} \in H_k$ for $k \geq 2$. Introduce new coordinates $y \in \mathbb{R}^n$ by the substitution

$$x = y + h^{(m)}(y), \tag{1.11}$$

where $h^{(m)} \in H_m$ for some fixed $m \geq 2$. At this moment, $h^{(m)}$ is an arbitrary function from H_m. Notice that the substitution (1.11) is close to the identity near the origin and thus invertible there, and the inverse transformation

$$y = x - h^{(m)}(x) + O(\|x\|^{m+1}) \tag{1.12}$$

is also smooth. In the new coordinates y, the map (1.10) has the form

$$y \mapsto Ay + \sum_{k=2}^{m-1} f^{(k)}(y) + \left[f^{(m)}(y) - (M_A h^{(m)})(y)\right] + O(\|y\|^{m+1}), \tag{1.13}$$

where the linear operator M_A is defined by the formula

$$(M_A h)(y) := h(Ay) - Ah(y). \tag{1.14}$$

If $h \in H_m$, then $M_A h \in H_m$ for all $m \geq 2$.

Notice that all terms of order less than m in (1.13) are the same as in (1.10), while the terms of order m have changed and differ from $f^{(m)}(y)$ by $-(M_A h^{(m)})(y)$. Now, we define the *linear homological equation* in H_m:

$$M_A h^{(m)} = f^{(m)}. \tag{1.15}$$

If $f^{(m)}$ belongs to the *range* $M_A(H_m)$ of M_A, then there is a solution $h^{(m)}$ to (1.15), meaning that there is a transformation (1.11) that eliminates all homogeneous terms of order m in (1.10). In general, however, $f^{(m)} = g^{(m)} + r^{(m)}$, where $g^{(m)} \in M_A(H_m)$, while $r^{(m)}$ belongs to a *complement* $\widetilde{H}_m$ to $M_A(H_m)$ in H_m. Therefore, only the $g^{(m)}$ part of $f^{(m)}$ can be eliminated from (1.10) by a transformation (1.11). The remaining $r^{(m)}$ terms are called the *resonant terms* of order m. Since $\widetilde{H}_m$ is not uniquely defined, the same is true for the resonant terms.

Cambridge University Press
978-1-108-49967-5 — Numerical Bifurcation Analysis of Maps
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Applying the above elimination procedure recursively for $m = 2, 3, 4, \ldots$, one proves the following theorem going back to Poincaré.

Theorem 1.5 (Poincaré Normal Form) *There is a polynomial change of coordinates*

$$x = y + h^{(2)}(y) + h^{(3)}(y) + \cdots + h^{(m)}(y),\ h^{(k)} \in H_k,$$

that transforms a smooth map

$$x \mapsto Ax + f(x), \quad x \in \mathbb{R}^n, \tag{1.16}$$

with $f(x) = O(\|x\|^2)$ *into*

$$y \mapsto Ay + r^{(2)}(y) + r^{(3)}(y) + \cdots + r^{(m)}(y) + O(\|y\|^{m+1}), \tag{1.17}$$

where each $r^{(k)}$ *contains only resonant terms of order* k*, i.e.,* $r^{(k)} \in \widetilde{H}_k$ *for* $k = 2, 3, \ldots, m$.

If all eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A are real and different, one can assume that A is diagonal, while the standard unit vectors $\{e_j\}_{j=1,2,\ldots,n}$ are the corresponding eigenvectors. In the space H_m, the operator M_A then has eigenvalues $(\lambda_1^{m_1}\lambda_2^{m_2}\cdots\lambda_n^{m_n} - \lambda_j)$, where $m_1 + m_2 + \cdots + m_n = m$. In this case, the homogeneous vector-monomials

$$x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}\, e_j$$

are the eigenvectors of M_A in H_m. If a *resonance* occurs, i.e.,

$$\lambda_j = \lambda_1^{m_1}\lambda_2^{m_2}\cdots\lambda_n^{m_n}$$

with $m_j \geq 0,\ m \geq 2$, the corresponding vector-monomial is not in the range of M_A and thus defines a resonant term. This allows determining resonant terms without long computations.

Note that all formulated results are also valid in the complex case, when $x, y \in \mathbb{C}^n$ and the complex matrix A has n different eigenvalues.

System (1.17) is called the *Poincaré normal form* of (1.16). In Chapter 4 we will give an efficient method to find coefficients of the normal forms of maps restricted to center manifolds, that combines the Poincaré normalization with the computation of the center manifold.

When considering a family of maps (1.8) depending on parameters, two approaches to its parameter-dependent normal forms are possible. One can try to find a normalizing transformation in $\mathbb{R}^n$ with coefficients that smoothly depend on parameters. Alternatively, one can consider the extended map (1.9) in the (x, α)-space and apply a normalization there. The former approach works well if the critical fixed point has a smooth continuation for nearby parameter values, i.e., there is no eigenvalue 1. The latter approach is necessary if such an eigenvalue is present.

1.4 Approximating ODEs

When dealing with local codim 2 bifurcations, we will repeatedly use the approximation of maps near their fixed points by shifts along orbits of certain systems of autonomous ordinary differential equations (ODEs). This allows us to predict *global* bifurcations of closed invariant curves and tori happening in the maps near cyclic, homo-, and heteroclinic bifurcations of the approximating ODEs. Although the exact bifurcation structure is *different* for maps and approximating ODEs, they provide information that is hardly available by analysis of the maps alone.

Consider a map having a fixed point $x = 0$:

$$x \mapsto f(x) = Ax + f^{(2)}(x) + f^{(3)}(x) + \cdots, \quad x \in \mathbb{R}^n, \tag{1.18}$$

where A is the Jacobian matrix of f at $x = 0$, while each component of $f^{(k)} \in H_k$ is a homogeneous polynomial of order k, $f^{(k)}(x) = O(\|x\|^k)$:

$$f_i^{(k)}(x) = \sum_{j_1+j_2+\cdots+j_n=k} b^{(k)}_{i,j_1 j_2 \cdots j_n} x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}.$$

In addition, consider a system of differential equations of the *same* dimension as the map (1.18) having an equilibrium at the point $x = 0$:

$$\dot{x} = F(x) = \Lambda x + F^{(2)}(x) + F^{(3)}(x) + \cdots, \quad x \in \mathbb{R}^n, \tag{1.19}$$

where Λ is a matrix and the terms $F^{(k)}$ have the same properties as the corresponding $f^{(k)}$ above. Denote by $\varphi^t(x)$ the (local) flow associated with (1.19). An interesting question is whether it is possible to construct a system (1.19), whose *unit-time shift* φ^1 along orbits coincides with (or at least approximates) the map f given by (1.18).

The map (1.18) is said to be *approximated up to order* k by system (1.19) if its Taylor expansion coincides with that of the unit-time shift φ^1 along the orbits of (1.19) up to and including terms of order k:

$$f(x) = \varphi^1(x) + O(\|x\|^{k+1}).$$

System (1.19) is then called an *approximating ODE system*.

We can construct the Taylor expansion of $\varphi^t(x)$ with respect to x at $x = 0$ as follows using *Picard iterations*. Namely, set

$$x^{(1)}(t) = e^{\Lambda t} x.$$