Cambridge University Press & Assessment
978-1-108-49960-6 — Intermediate Solid Mechanics
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1 Analysis of Stress

The concept of stress is the most fundamental concept in the mechanics of solids. The discrete atomistic structure of a material is ignored and the model of a continuum is adopted, according to which the entire space between the boundaries of a considered body is filled with the material. If, at a considered point of a solid body, an infinitesimal surface element $\mathrm{d}S$, with a unit outward normal vector $\mathbf{n}$, transmits a force $\mathrm{d}\mathbf{F}_n$, the traction vector at that point with respect to the considered surface element is defined by the ratio $\mathbf{t}_n = \mathrm{d}\mathbf{F}_n/\mathrm{d}S$. The projection of the traction vector in the direction of the vector orthogonal to the surface element is the normal stress over that surface element, $\sigma_{nn} = \mathbf{t}_n \cdot \mathbf{n}$. The remaining component, tangential to the surface element, is the shear stress $\sigma_{nm}\mathbf{m} = \mathbf{t}_n - \sigma_{nn}\mathbf{n}$, where $\mathbf{m}$ is a unit vector tangential to $\mathrm{d}S$. The first index (n) in the stress component σ_{nm} specifies the orientation of the surface element over which σ_{nm} acts, i.e., the direction of the unit vector orthogonal to $\mathrm{d}S$, while the second index (m) specifies the direction tangential to $\mathrm{d}S$ along which the stress component σ_{nm} physically acts. This chapter is devoted to the analysis of the normal and shear stresses over differently oriented surface elements through a considered material point of a loaded body. The analysis leads to the notion of a stress tensor, originally introduced by the French mathematician, physicist, and engineer Augustin-Louis Cauchy in the nineteenth century. We present the analysis of one-, two-, and three-dimensional states of stress, determine the corresponding principal stresses (maximum and minimum normal stresses) and the maximum shear stress, define the deviatoric and spherical parts of the stress tensor, derive the equations of equilibrium which must be satisfied by the stress field within a loaded body at rest, and formulate the corresponding boundary conditions.

1.1 Traction Vector

At any point of a loaded body, the traction vector $\mathbf{t}_n$, relative to a surface element $\mathrm{d}S$ whose unit normal vector is $\mathbf{n}$ (Fig. 1.1), is defined such that

$$\mathbf{t}_n = \frac{\mathrm{d}\mathbf{f}_n}{\mathrm{d}S}, \tag{1.1}$$

where $\mathrm{d}\mathbf{f}_n$ is the force transmitted by $\mathrm{d}S$. Figure 1.2(a) shows traction vectors acting on four sides of an infinitesimal rectangular material element with sides $\mathrm{d}x$ and $\mathrm{d}y$, having

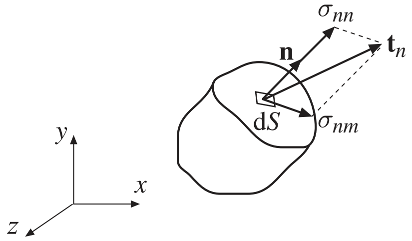
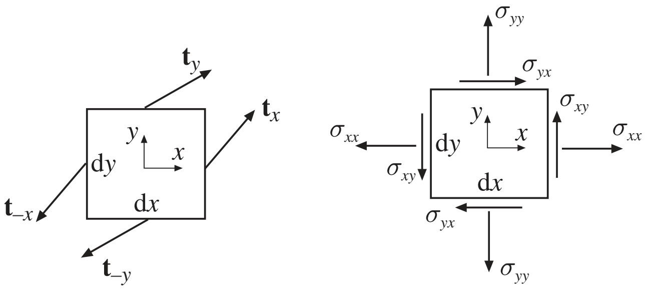


Figure 1.1 Traction vector $\mathbf{t}_n$ over the surface element $\mathrm{d}S$ with unit normal vector $\mathbf{n}$, and its normal and shear stress components (σ_{nn} and σ_{nm}).



(a) (b)

Figure 1.2 (a) Traction vectors $\mathbf{t}_{\pm x}$ and $\mathbf{t}_{\pm y}$ over four sides of a uniformly stressed rectangular material element $(\mathrm{d}x, \mathrm{d}y)$. By the law of action and reaction, $\mathbf{t}_{-x} = -\mathbf{t}_x$ and $\mathbf{t}_{-y} = -\mathbf{t}_y$. (b) The normal and shear stress components of the traction vectors from part (a).

a unit thickness in the z direction. In this two-dimensional case, the traction vectors over the sides whose unit normal vectors are in the positive x and y directions have the normal and shear stress components

$$\mathbf{t}_x = \{\sigma_{xx}, \sigma_{xy}\}, \quad \mathbf{t}_y = \{\sigma_{yx}, \sigma_{yy}\}. \tag{1.2}$$

The normal stresses acting on the two planes are σ_{xx} and σ_{yy}, while the accompanying shear stresses are σ_{xy} and σ_{yx} (Fig. 1.2(b)). The ordering of the indices is such that, for example, σ_{yx} represents a stress component in the x direction, acting over the surface element whose normal vector is in the y direction. In other words, the first index specifies the direction of the surface normal vector, while the second index specifies the direction of the stress component itself. By Newton's third law of action and reaction, or by equilibrium of an infinitesimally thin slice of material whose sides have unit normals in the x and $-x$ directions, it follows that $\mathbf{t}_{-x} = -\mathbf{t}_x$ and $\mathbf{t}_{-y} = -\mathbf{t}_y$. Thus, the directions of the positive stress components are as shown in Fig. 1.2(b), i.e., if the normal vector to the surface element is in the positive coordinate direction, the positive stress components act in the positive coordinate directions, and vice versa.

The shear stresses σ_{xy} and σ_{yx} obey the conjugacy property

$$\sigma_{xy} = \sigma_{yx}, \tag{1.3}$$

which follows from the moment equilibrium condition (Fig. 1.2(b))

$$\sum M_z = 0: \quad (\sigma_{xy}\mathrm{d}y \cdot 1)\mathrm{d}x - (\sigma_{yx}\mathrm{d}x \cdot 1)\mathrm{d}y = 0. \tag{1.4}$$

The stress matrix (with respect to coordinate axes x and y) is obtained by grouping the two traction vectors, as follows

$$\begin{bmatrix} \mathbf{t}_x \\ \mathbf{t}_y \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix}. \tag{1.5}$$

The three-dimensional generalizations of (1.2) and (1.5) are (Fig. 1.3)

$$\mathbf{t}_x = \{\sigma_{xx}, \sigma_{xy}, \sigma_{xz}\}, \quad \mathbf{t}_y = \{\sigma_{yx}, \sigma_{yy}, \sigma_{yz}\}, \quad \mathbf{t}_z = \{\sigma_{zx}, \sigma_{zy}, \sigma_{zz}\}, \tag{1.6}$$

and

$$\begin{bmatrix} \mathbf{t}_x \\ \mathbf{t}_y \\ \mathbf{t}_z \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}. \tag{1.7}$$

The corresponding conjugacy properties of shear stresses are

$$\sigma_{xy} = \sigma_{yx}, \quad \sigma_{yz} = \sigma_{zy}, \quad \sigma_{zx} = \sigma_{xz}, \tag{1.8}$$

which follow from the moment equilibrium conditions for the z, x, and y axis, respectively, and which make the stress matrix in (1.7) symmetric.

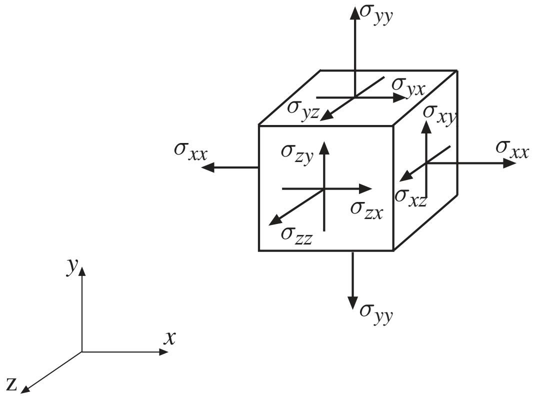


Figure 1.3 Rectangular stress components of a three-dimensional stress.

1.2 Cauchy Relation for Traction Vectors

If traction vectors in three orthogonal planes through a considered point are known, say $(\mathbf{t}_x, \mathbf{t}_y, \mathbf{t}_z)$, the traction vector in any other plane through that point is also known and given by the Cauchy relation

$$\mathbf{t}_n = n_x \mathbf{t}_x + n_y \mathbf{t}_y + n_z \mathbf{t}_z, \tag{1.9}$$

where $\mathbf{n} = \{n_x, n_y, n_z\}$ is the unit vector orthogonal to the considered plane (Fig. 1.4).

For simplicity, we prove below the Cauchy relation in the case of two-dimensional state of stress (Fig. 1.5(a)). The free-body diagram of an extracted triangular material element is shown in Fig. 1.5(b). The unit vector orthogonal to the inclined plane is $\mathbf{n} = \{n_x, n_y\}$, where $n_x = \cos\varphi$ and $n_y = \sin\varphi$. For equilibrium, the sum of all forces (per unit length in the z direction) must vanish,

$$\mathbf{t}_{-x}\mathrm{d}y + \mathbf{t}_{-y}(\mathrm{d}y \tan\varphi) + \mathbf{t}_n(\mathrm{d}y/\cos\varphi) + \mathbf{b}(\mathrm{d}y)^2(\tan\varphi)/2 = \mathbf{0}. \tag{1.10}$$

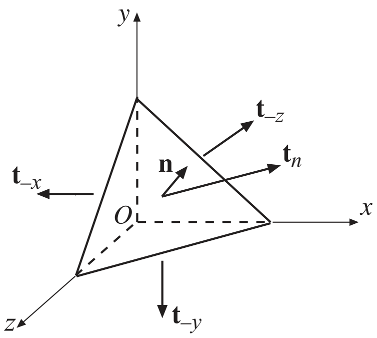
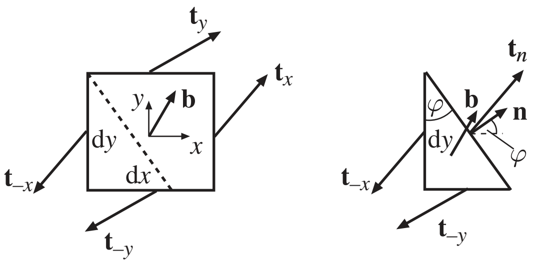


Figure 1.4 A tetrahedron material element with traction vectors on its four sides. The traction vector on the inclined plane with unit normal vector $\mathbf{n}$ is $\mathbf{t}_n$.



(a) (b)

Figure 1.5 (a) Traction vectors $\mathbf{t}_{\pm x}$ and $\mathbf{t}_{\pm y}$ over four sides of the rectangular material element $(\mathrm{d}x, \mathrm{d}y)$. The body force per unit volume is $\mathbf{b}$. (b) The free-body diagram of a triangular element extracted from the rectangular element of part (a) along an inclined plane whose normal vector is $\mathbf{n} = \{n_x, n_y\} = \{\cos\varphi, \sin\varphi\}$.

The body force (per unit volume) is **b**, and the area of the triangular element is equal to $(\mathrm{d}y)^2(\tan\varphi)/2$. Upon dividing (1.10) by $\mathrm{d}y$ and performing the limit $\mathrm{d}y \to 0$, it follows that

$$\mathbf{t}_n = (\cos\varphi)\mathbf{t}_{-x} + (\sin\varphi)\mathbf{t}_{-y}. \tag{1.11}$$

Since $\mathbf{t}_{-x} = -\mathbf{t}_x$, $\mathbf{t}_{-y} = -\mathbf{t}_y$, $n_x = \cos\varphi$, and $n_y = \sin\varphi$, (1.11) becomes

$$\mathbf{t}_n = n_x\mathbf{t}_x + n_y\mathbf{t}_y, \tag{1.12}$$

which is a two-dimensional version of the Cauchy relation (1.9).

The matrix representation of the Cauchy relation (1.9) is $\mathbf{t}_n = [\sigma]\cdot\mathbf{n}$, i.e., in expanded form,

$$\begin{bmatrix} t_{nx} \\ t_{ny} \\ t_{nz} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \cdot \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}. \tag{1.13}$$

Thus, the (x, y, z) components of the traction vector $\mathbf{t}_n$ are

$$\begin{aligned} t_{nx} &= \sigma_{xx}n_x + \sigma_{xy}n_y + \sigma_{xz}n_z, \\ t_{ny} &= \sigma_{yx}n_x + \sigma_{yy}n_y + \sigma_{yz}n_z, \\ t_{nz} &= \sigma_{zx}n_x + \sigma_{zy}n_y + \sigma_{zz}n_z. \end{aligned} \tag{1.14}$$

Note that in the matrix representation (1.13), the vectors $\mathbf{t}_n$ and $\mathbf{n}$ are considered to be column vectors of dimension (3×1).

Example 1.1 Prove the conjugacy relation for traction vectors $\mathbf{m}\cdot\mathbf{t}_n = \mathbf{n}\cdot\mathbf{t}_m$, where **m** and **n** are two, not necessarily orthogonal, unit vectors.

Solution

By the matrix form of the Cauchy relation (1.13), we can write $\mathbf{t}_n = [\sigma]\cdot\mathbf{n}$ and $\mathbf{t}_m = [\sigma]\cdot\mathbf{m}$. Thus, $\mathbf{m}\cdot\mathbf{t}_n = \mathbf{m}^{\mathrm{T}}\cdot[\sigma]\cdot\mathbf{n}$ and $\mathbf{n}\cdot\mathbf{t}_m = \mathbf{n}^{\mathrm{T}}\cdot[\sigma]\cdot\mathbf{m}$, where $()^{\mathrm{T}}$ denotes the transpose, so that $\mathbf{m}^{\mathrm{T}}$ and $\mathbf{n}^{\mathrm{T}}$ are row vectors. Since $[\sigma]$ is a symmetric matrix, it follows (by the definition of symmetric matrices) that $\mathbf{m}^{\mathrm{T}}\cdot[\sigma]\cdot\mathbf{n} = \mathbf{n}^{\mathrm{T}}\cdot[\sigma]\cdot\mathbf{m}$, for any two vectors **m** and **n**. Thus, $\mathbf{m}\cdot\mathbf{t}_n = \mathbf{n}\cdot\mathbf{t}_m$.

1.3 Normal and Shear Stresses over an Inclined Plane

The normal stress over an inclined plane is obtained by projecting the traction vector $\mathbf{t}_n$ onto the normal vector **n** (Fig. 1.4),

$$\sigma_{nn} = \mathbf{n}\cdot\mathbf{t}_n = n_x t_{nx} + n_y t_{ny} + n_z t_{nz}. \tag{1.15}$$

Substituting (1.14) into (1.15) gives

$$\sigma_{nn} = \sigma_{xx}n_x^2 + \sigma_{yy}n_y^2 + \sigma_{zz}n_z^2 + 2(\sigma_{xy}n_xn_y + \sigma_{yz}n_yn_z + \sigma_{zx}n_zn_x). \tag{1.16}$$

The matrix representation of (1.15) is

$$\sigma_{nn} = \begin{bmatrix} n_x & n_y & n_z \end{bmatrix} \cdot \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \cdot \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}. \tag{1.17}$$

The traction vector in an inclined plane can be decomposed into its normal and shear stress components, such that

$$\mathbf{t}_n = \sigma_{nn}\mathbf{n} + \sigma_{nm}\mathbf{m}, \tag{1.18}$$

where $\mathbf{m}$ is a unit vector within the plane under consideration, and σ_{nm} is the corresponding shear stress in that plane. The magnitude of σ_{nm} can be obtained from Pythagoras' theorem as

$$\sigma_{nm} = (t_n^2 - \sigma_{nn}^2)^{1/2} = (t_{nx}^2 + t_{ny}^2 + t_{nz}^2 - \sigma_{nn}^2)^{1/2}, \tag{1.19}$$

where (t_{nx}, t_{ny}, t_{nz}) are given by (1.14), and σ_{nn} by (1.16).

1.3.1 Two-Dimensional State of Stress

In the two-dimensional case, $\sigma_{zx} = \sigma_{zy} = \sigma_{zz} = 0$ and the expression (1.16) reduces to

$$\sigma_{nn} = \sigma_{xx}n_x^2 + \sigma_{yy}n_y^2 + 2\sigma_{xy}n_x n_y. \tag{1.20}$$

This can also be obtained directly by projecting the traction vector $\mathbf{t}_n$, as given by (1.12), in the direction $\mathbf{n} = \{n_x, n_y\}$ (Fig. 1.6).

Since $n_x = \cos\varphi$ and $n_y = \sin\varphi$, (1.20) can be rewritten as

$$\sigma_{nn} = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) + \frac{1}{2}(\sigma_{xx} - \sigma_{yy})\cos 2\varphi + \sigma_{xy}\sin 2\varphi, \tag{1.21}$$

with the trigonometric identities

$$\cos^2\varphi = \frac{1}{2}(1 + \cos 2\varphi), \quad \sin^2\varphi = \frac{1}{2}(1 - \cos 2\varphi) \tag{1.22}$$

conveniently used.

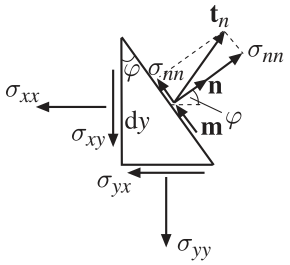


Figure 1.6 The traction vector $\mathbf{t}_n$ acting over an inclined plane whose unit normal vector is $\mathbf{n}$, with its normal and shear stress components (σ_{nn} and σ_{nm}).

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The shear stress σ_{nm} is obtained by projecting the traction vector $\mathbf{t}_n$ from (1.12) in the direction $\mathbf{m} = \{m_x, m_y\}$ (Fig. 1.6), i.e., $\sigma_{nm} = \mathbf{m} \cdot \mathbf{t}_n$. This gives

$$\sigma_{nm} = \sigma_{xx} n_x m_x + \sigma_{yy} n_y m_y + \sigma_{xy} n_x m_y + \sigma_{yx} n_y m_x. \tag{1.23}$$

Since $m_x = -n_y$ and $m_y = n_x$, the above can be rewritten as

$$\sigma_{nm} = -(\sigma_{xx} - \sigma_{yy}) n_x n_y + \sigma_{xy}(n_x^2 - n_y^2), \tag{1.24}$$

or

$$\sigma_{nm} = -\frac{1}{2}(\sigma_{xx} - \sigma_{yy}) \sin 2\varphi + \sigma_{xy} \cos 2\varphi. \tag{1.25}$$

1.4 Tensorial Nature of Stress

By (1.20) and (1.23), we have in the two-dimensional case

$$\begin{aligned}
\sigma_{nn} &= \sigma_{xx} n_x^2 + \sigma_{yy} n_y^2 + 2\sigma_{xy} n_x n_y, \\
\sigma_{mm} &= \sigma_{xx} m_x^2 + \sigma_{yy} m_y^2 + 2\sigma_{xy} m_x m_y, \\
\sigma_{nm} &= \sigma_{xx} n_x m_x + \sigma_{yy} n_y m_y + \sigma_{xy} n_x m_y + \sigma_{yx} n_y m_x, \\
\sigma_{mn} &= \sigma_{xx} m_x n_x + \sigma_{yy} m_y n_y + \sigma_{xy} m_x n_y + \sigma_{yx} m_y n_x.
\end{aligned} \tag{1.26}$$

This can be cast in the matrix form

$$\begin{bmatrix} \sigma_{nn} & \sigma_{nm} \\ \sigma_{mn} & \sigma_{mm} \end{bmatrix} = \begin{bmatrix} n_x & m_x \\ n_y & m_y \end{bmatrix}^{\mathrm{T}} \cdot \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \cdot \begin{bmatrix} n_x & m_x \\ n_y & m_y \end{bmatrix}, \tag{1.27}$$

where T denotes the transpose, and

$$\begin{bmatrix} n_x & m_x \\ n_y & m_y \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \tag{1.28}$$

is the rotation matrix that transforms the (x, y) components of any vector $\mathbf{r}$ into its (n, m) components

$$\begin{bmatrix} r_n \\ r_m \end{bmatrix} = \begin{bmatrix} n_x & m_x \\ n_y & m_y \end{bmatrix} \cdot \begin{bmatrix} r_x \\ r_y \end{bmatrix}. \tag{1.29}$$

When the components of a 2 × 2 matrix transform under the rotation of the coordinate system according to (1.27), the matrix is said to be a second-order tensor.

A three-dimensional stress, represented by a 3 × 3 matrix appearing in (1.7), is a second-order tensor, because it obeys the transformation rule of the type (1.27), with the rotation tensor

$$[Q] = \begin{bmatrix} n_x & m_x & s_x \\ n_y & m_y & s_y \\ n_z & m_z & s_z \end{bmatrix}, \tag{1.30}$$

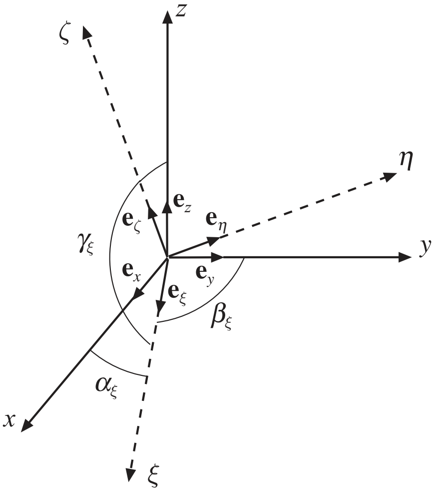

Figure 1.7 The coordinate system (ξ, η, ζ) obtained by the rotation from the coordinate system (x, y, z). The unit vectors of the two sets of coordinate axes are $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ and $(\mathbf{e}_\xi, \mathbf{e}_\eta, \mathbf{e}_\zeta)$.

where (**n**, **m**, **s**) are the unit vectors of the rotated coordinate system. Thus, if the new coordinate axes are (ξ, η, ζ), we have

$$[\sigma]_{(\xi,\eta,\zeta)} = [Q]^{\mathrm{T}} \cdot [\sigma]_{(x,y,z)} \cdot [Q]. \tag{1.31}$$

Geometrically, if the ξ axis makes the angles $(\alpha_\xi, \beta_\xi, \gamma_\xi)$ with the positive x axis (Fig. 1.7), then the unit vector **n** along the positive ξ axis is

$$\mathbf{n} = \mathbf{e}_\xi = \{\cos\alpha_\xi, \cos\beta_\xi, \cos\gamma_\xi\}. \tag{1.32}$$

Similarly, for the unit vectors **m** and **s** along the positive η and ζ axes,

$$\mathbf{m} = \mathbf{e}_\eta = \{\cos\alpha_\eta, \cos\beta_\eta, \cos\gamma_\eta\}, \quad \mathbf{s} = \mathbf{e}_\zeta = \{\cos\alpha_\zeta, \cos\beta_\zeta, \cos\gamma_\zeta\}. \tag{1.33}$$

The rotation matrix in (1.30) is thus

$$[Q] = \begin{bmatrix} \cos\alpha_\xi & \cos\alpha_\eta & \cos\alpha_\zeta \\ \cos\beta_\xi & \cos\beta_\eta & \cos\beta_\zeta \\ \cos\gamma_\xi & \cos\gamma_\eta & \cos\gamma_\zeta \end{bmatrix}. \tag{1.34}$$

1.5 Principal Stresses: 2D State of Stress

For design purposes it is of fundamental importance to determine the maximum normal stress at a considered point of a loaded body. Considering first a two-dimensional state of stress, the normal stress on an inclined plane whose unit normal vector is **n** is given by (1.20), i.e.,

$$\sigma_{nn} = \sigma_{xx} n_x^2 + \sigma_{yy} n_y^2 + 2\sigma_{xy} n_x n_y. \tag{1.35}$$

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The objective is to find the plane (specified by the components n_x and n_y of its normal vector), over which σ_{nn} is maximum (or minimum). Since n_x and n_y are the components of a unit vector, we search for an extreme value of the function $\sigma_{nn}(n_x, n_y)$ subject to the constraint $n_x^2 + n_y^2 = 1$. Thus, we introduce the Lagrangian multiplier σ and search for an unconstrained extremum of the function $f = \sigma_{nn} - \sigma(n_x^2 + n_y^2 - 1)$, i.e.,

$$f = \sigma_{xx}n_x^2 + \sigma_{yy}n_y^2 + 2\sigma_{xy}n_x n_y - \sigma(n_x^2 + n_y^2 - 1). \tag{1.36}$$

The stationarity conditions for f are

$$\frac{\partial f}{\partial n_x} = 0, \quad \frac{\partial f}{\partial n_y} = 0, \tag{1.37}$$

which give

$$\begin{aligned} (\sigma_{xx} - \sigma)n_x + \sigma_{xy}n_y &= 0, \\ \sigma_{xy}n_x + (\sigma_{yy} - \sigma)n_y &= 0. \end{aligned} \tag{1.38}$$

We next prove that σ is in fact the maximum or minimum normal stress. By multiplying the first equation in (1.38) by $\mathbf{e}_x$ and the second by $\mathbf{e}_y$, where $\mathbf{e}_x$ and $\mathbf{e}_y$ are the unit vectors along the x and y axes, by adding up the resulting two expressions, and by using (1.2), the traction vector in the principal plane is found to be

$$\mathbf{t}_n = n_x\mathbf{t}_x + n_y\mathbf{t}_y = \sigma\mathbf{n}. \tag{1.39}$$

Thus, $\mathbf{t}_n$ in the principal plane is entirely in the $\mathbf{n}$ direction, having no shear component at all, and σ represents the (extreme) normal stress in that plane.

Since $\mathbf{t}_n = [\sigma] \cdot \mathbf{n}$, (1.39) can be rewritten as

$$[\sigma] \cdot \mathbf{n} = \sigma\mathbf{n}, \tag{1.40}$$

which means that σ and $\mathbf{n}$ satisfying (1.40) are the eigenvalues and eigenvectors of the stress tensor $[\sigma]$.

To determine σ, we use the fact that (1.38) is a homogeneous system of two linear algebraic equations for n_x and n_y, which has a solution if and only if the determinant of the system vanishes,

$$\begin{vmatrix} \sigma_{xx} - \sigma & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} - \sigma \end{vmatrix} = 0. \tag{1.41}$$

Upon expansion, (1.41) yields a quadratic equation for σ,

$$\sigma^2 - I_1\sigma - I_2 = 0, \tag{1.42}$$

where

$$I_1 = \sigma_{xx} + \sigma_{yy}, \quad I_2 = -\sigma_{xx}\sigma_{yy} + \sigma_{xy}^2 \tag{1.43}$$

are two invariants of the 2×2 stress tensor $[\sigma]$. These are called invariants because they have the same values regardless of the coordinate system used to express the stress tensor, i.e., the (x, y) coordinate system or any other coordinate system obtained from the (x, y) system by rotation. Physically, the principal stress σ cannot depend on the

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coordinate system used to express the stress tensor, and thus the coefficients I_1 and I_2 in equation (1.42) cannot depend on the coordinate system either.

The solution of the quadratic equation (1.42) is

$$\sigma_{1,2} = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) \pm \frac{1}{2}\left[(\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2\right]^{1/2}. \tag{1.44}$$

These are the so-called principal stresses; $\sigma_1 = \sigma_{\max}$ is the maximum normal stress, and $\sigma_2 = \sigma_{\min}$ is the minimum normal stress. The corresponding directions are the principal directions (1) and (2). They are specified by either the first or second equation in (1.38). For example, from the first equation it follows that

$$\tan\varphi_{1,2} = \left(\frac{n_y}{n_x}\right)_{1,2} = \frac{\sigma_{1,2} - \sigma_{xx}}{\sigma_{xy}}. \tag{1.45}$$

Upon the substitution of (1.44), this becomes

$$\tan\varphi_{1,2} = -\frac{\sigma_{xx} - \sigma_{yy}}{2\sigma_{xy}} \pm \frac{\left[(\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2\right]^{1/2}}{2\sigma_{xy}}. \tag{1.46}$$

It can be readily verified that

$$\tan\varphi_1 \cdot \tan\varphi_2 = -1, \tag{1.47}$$

which implies that the angles φ_1 and φ_2 are 90° apart. Also, since $\sigma_1 > \sigma_{xx}$, it follows from (1.45) that $\tan\varphi_1 > 0$ (i.e., $0^\circ < \varphi_1 < 90^\circ$) if $\sigma_{xy} > 0$. If $\sigma_{xy} < 0$, then $-90^\circ < \varphi_1 < 0^\circ$.

Alternatively, by multiplying the first equation of (1.38) by n_y and the second by n_x, and subtracting the resulting two expressions, we obtain

$$\tan 2\varphi = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}}, \tag{1.48}$$

whose two solutions give the angles $\varphi_{1,2}$ in accord with (1.46). A rectangular material element with the sides along the principal directions is shown in Fig. 1.8(a).

Exercise 1.1 Derive (1.48) directly from (1.21) by finding its extremum from the condition $\partial\sigma_{nn}/\partial\varphi = 0$.

Exercise 1.2 The shear stress in the principal plane is always zero. Prove that $\sigma_{nm} = 0$ in the principal plane by substituting (1.48) into (1.25).

1.6 Maximum Shear Stress: 2D Case

For ductile materials it is of particular importance to determine the maximum shear stress at any point of a loaded body, because large shear stresses can cause plastic deformation. The shear stress on an inclined plane whose unit normal vector is **n** is given by (1.24), i.e.,

$$\sigma_{nm} = -(\sigma_{xx} - \sigma_{yy})n_x n_y + \sigma_{xy}(n_x^2 - n_y^2). \tag{1.49}$$