1. The Physical Grounds of Radiative Transfer

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Abstract

Radiative transfer, i.e., the transport of radiant energy through a medium, can be described in several alternative ways, either at macroscopic or microscopic level. In order to set a common physical background for the applications of radiative transfer to stellar and planetary atmospheres, presented in the second part of this book, a macroscopic representation of the radiation field derived from radiometry, a microscopic picture based on the kinetics of photons and the transport of radiant energy in terms of Maxwell’s electromagnetic theory are discussed.

1.1 Introduction

The aim of this chapter is to define a common physical framework, established by radiative transfer, for the ensuing discussion of the phenomenology of early-type, late-type and planetary atmospheres. Radiative transfer (RT), i.e., the transport of radiant energy, is a key phenomenon in astrophysics and plays a central role in this School.

Radiant energy may be defined as energy in form of electromagnetic waves that propagate either in a vacuum or through a material medium. In the latter case, the radiation field can exchange both energy and momentum with matter. In order to account for the transport of radiant energy, three different equivalent representations can be introduced: (a) the electrodynamic picture according to Maxwell’s equations and the Poynting vector (waves); (b) the description in terms of a continuous stream of energy in given directions (rays); and (c) the corpuscular picture, revisited after Einstein’s hypothesis of light quantum (flow of photons). The single underlying concept is that of radiant energy in motion, which is a transport process. This gives rise in a natural way to the concept of radiant flux, namely, the amount of radiant energy flowing per unit time across a given surface.

Newton’s corpuscular hypothesis and the consequent light ray model (Newton, 1704) superseded Huygens’s earlier wave theory (Huygens, 1690); Young’s interference experiments in 1801 subsequently vindicated Huygens’s theory. In 1873, Maxwell achieved the unification of optics and electrodynamics in terms of electromagnetic waves (Maxwell, 1891), later confirmed experimentally by Hertz. Einstein’s interpretation of the photoelectric effect set the ball rolling again. Nowadays, quantum physics postulates the dual wave–particle nature of electromagnetic radiation so that its transport is ascribed to either of two different kinds of carriers: electromagnetic waves and photons. Only quantum electrodynamics, where the electromagnetic field is represented by operators and the radiation field by quantized harmonic oscillators, can the properties of photons be rigorously accounted for. Nevertheless, in our exposition we will assume a semiclassical point of view, according to which photons are considered localized particlelike entities with well-defined energy and momentum.

The solution of the Maxwell’s equations implies the transport of energy through the surrounding medium by means of electromagnetic waves that propagate with a finite speed characteristic of the specific medium. An alternative phenomenological picture may...
be drawn from radiometric concepts, based on the single physical idea of radiant flux. In order to develop suitable mathematical tools for this representation, the first necessary step is to define, on a sound physical basis, the entity transported. Two alternative (but correlated) approaches are possible in terms of either macroscopic or microscopic quantities.

After introducing in Section 1.2 the stellar atmosphere physical system and a brief mention of rays and geometrical optics in Section 1.3, we will recall the basic concept of radiometry (Section 1.4), prior to an operational definition of the macroscopic specific intensity of the radiation field that is given in Section 1.5. An alternative microscopic picture will be introduced in Section 1.6 in order to derive in Section 1.7 the RT equation as a kinetic (Boltzmann’s) equation for the photons. Macroscopic RT coefficients are defined operationally in Section 1.8, which will allow us to analyse in Section 1.9 the structure of the source and sink terms in the RT equation, and to define the source function through which the specific RT equations are coupled. The statistical interpretation of radiative transfer is considered in Section 1.10. The transport of radiant energy is described as a fluid dynamics-like process in Section 1.11, which will serve as an introduction to Section 1.13, where the macroscopic and the electrodynamic pictures (Section 1.12) of the radiation field are compared.

1.2 The Stellar Atmosphere Physical System

1.2.1 Definition of a Star

A star may be defined as a gravitationally bound open concentration of matter and energy. From the point of view of thermodynamics, which is essentially concerned with the flow and balance of energy and matter, physical systems may be classified as:

open, when both matter and energy fluxes are present;

closed, when only energy fluxes occur; and

isolated, in which neither matter nor energy fluxes are present

The mere fact that we see the stars is clear-cut evidence that they emit radiant energy over a vast wavelength range. Moreover, the long-standing detection of the so-called solar and stellar winds has revealed important mass-loss phenomena, hence the justification of the previous definition.

The observed evidence of fluxes implies the existence of gradients in the physical properties inside the star and hence of transport phenomena that tend to establish equilibrium conditions. The energy generated by thermonuclear reactions in the inner core is carried through the stellar layers by two modes of transport: radiative and convective. Their relative weight depends on the point-by-point thermodynamic conditions inside the star. However, the primary observational fact that stars emit radiation is clear-cut proof that the former must always be present. Radiation pressure, acting outwards, is antagonist to the gravitational force. The balance between the force due to the total pressure (gas plus radiation) and gravity keeps the structure of the star stationary over very long periods of its life. All this characterizes a system that is out of equilibrium where irreversible processes are taking place. Since the system is not far from equilibrium, however, the transport phenomena are governed by linear phenomenological laws. Thus, to a first approximation, a linear nonequilibrium approach may be employed.

¹ Macroscopic quantities are suggested directly by experience, without any preliminary knowledge of their intrinsic nature. In contrast, a microscopic picture requires previous hypotheses concerning the entity to be represented.
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1.2.2 Qualitative Definition of a Stellar Atmosphere

The outermost layers of a star, i.e., its atmosphere, constitute a boundary through which photons can escape in the form of radiation into the surrounding interstellar medium. We will therefore define the stellar atmosphere to be this frontier region, where the emergent electromagnetic spectrum forms. The radiative flux observed is the signature of gradients inside the atmosphere, which are in turn the consequence of departure from equilibrium conditions. Moreover, the evidence of mass loss from observations of solar and stellar winds shows that matter particles can also escape outwards, a proof of departure from mechanical equilibrium and hence the transport of matter. On the other hand, departure from radiative equilibrium in the atmosphere of late-type stars gives rise to convective transport, which plays a fundamental role in the energy balance, as well as in the generation and advection of the magnetic fields observed in these stars.

The stellar atmosphere physical system consists of two components, matter and a radiation field that permeates and interacts with the former. At the macroscopic level, its structure will be described at any point by the values of the thermodynamic variables $T$, $P$, and $\rho$, as well as the velocity of the (ideal) matter elements on the one hand, and by the local properties of the radiation field on the other. As in any physical system, the structure of the atmosphere is shaped by the mutual interactions of the two components, which are determined by the physical relations among the variables, the internal energy of the system and the constraints imposed. Such relations will be expressed by conservation, state and transport equations. We note that the structure of a stellar atmosphere is mainly determined by the physical properties of the stellar interior, most importantly by the temperature gradient of the whole star. This gradient is responsible for the outward radiative flux. The outer layers do not quantitatively alter the outgoing flux. Owing to their low density, they cannot absorb and store a large amount of energy, neither is the energy produced inside them comparable to the energy generated in the stellar interior.

The physical conditions inside a stellar atmosphere are governed essentially by the gravitational field generated by the star and the outward radiation flux from the interior. In order to remain in a steady state, the configuration of the atmosphere must be such that radiation can flows outwards. Therefore, two external parameters – the gravitational acceleration at the surface and the total radiation flux (i.e., the bolometric luminosity of the star) – together with an internal parameter, namely the chemical composition, determine the state of the stellar atmosphere physical system.

1.2.3 The Observational Side

From an observational point of view, the spectral features of the emergent radiation, namely the qualitative properties of this flux, are determined by interactions between matter and radiation in the outer layers. These processes are responsible for a redistribution in frequency of the radiant energy. The emergent spectrum therefore reflects the physical state of the stellar atmosphere, where, by definition, it forms. This statement constitutes the key to the diagnostics of the physical properties of stars: from the theoretical modelling of stellar atmospheres, we try to predict the characteristics of the emergent spectrum via the computation of the radiative flux carried through the outer layers; we then compare successively the computed with the observed features.

Looking forwards to the lectures that follow, we mention just two examples:

1. The supersonic velocity fields in early-type stars are revealed by line shifts and P Cyg profiles, which provide diagnostic tools to infer the mechanical structure of the expanding outer layers of these stars. Moreover, velocity fields bear upon radiative transfer in spectral lines via the Doppler effect.
2. The observation of asymmetries in the profiles of late-type giant stars suggests a departure from hydrostatic equilibrium (or steady state) in their atmospheres.

1.3 Rays and Geometrical Optics

1.3.1 Light Rays

Against the background of the seventeenth-century atomistic view of natural philosophy, Newton, going beyond the experimental evidence of his exhaustive investigations, put forward the conjecture that the geometrical behaviour of reflection and refraction could be explained only if light were made of corpuscles. The propagation of radiant energy as a flow of particles entails the ray concept. In his *Opticks*, Newton states: ‘The least Light, or part of Light, which may be stopp’d alone without the rest of the Light, or propagated alone, or do suffer anything alone, which the rest of the Light doth not or suffers not, I call a Ray of Light.’ The geometrical nature of the light rays is affirmed in the further definition: ‘Mathematicians usually consider the Rays of Light to be Lines reaching from the luminous Body to the Body illuminated.’

To some extent, however, a ray can be realized physically by allowing light from a distant source to pass through a small circular aperture of radius \( r \) pierced into a screen. When \( r \to 0 \), the tube of light emerging from the screen shrinks to a curve. Therefore, we may assume that the radiation field consists of an ensemble of rays, characterized by their direction and the amount of energy they carry. Hence, the transport of radiant energy from one point of the medium to another can be formulated in terms of the creation, propagation and destruction of rays.

1.3.2 Geometrical Optics

The ordinary properties of light, such as rectilinear propagation and reflection, or refraction at the interface between two material media, can be understood simply by knowing how light travels, without enquiring into its nature. In contrast with physical optics, which takes into account phenomena such as interference, diffraction and polarization, the foregoing simplified approach constitutes what is called geometrical (or ray) optics, an idealized model of light propagation in terms of rays. Because the wavelengths considered are very small in comparison with the spatial variation of any property of the medium through which the waves propagate, the value of the wavelength can be formally allowed to tend to zero. Hence, the laws of propagation, which determine the trajectories of the rays, have an essentially geometrical character.

In order to show that, under the necessary condition that the medium be isotropic, the rays are always perpendicular to the light wavefronts, it is necessary to make reference here to the Hamilton–Jacobi theory. As is well known, in this formulation of mechanics the motion of a particle (or a system of particles) is represented as a wave. The solution, \( S \), of the equation \( H + \partial S/\partial t = 0 \), where \( H \) is the Hamiltonian of the system, is called Hamilton’s principal function. For conservative systems, the motion of \( S \) in time is similar to the propagation of a wavefront in the configuration space. In the specific instance of light waves, the corresponding scalar equation is satisfied by a plane wave if the refraction index, \( n \), is constant (otherwise, variation of the latter will distort and bend the wave). When \( n \) does not vary greatly over distances of the order of the wavelength, which is the

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2 The first book of *Opticks*, Defs I and II, Part I.
3 This, for example, is the point of view of Planck (1906).
4 For a discussion of the motion of \( S \) in the configuration space, see, for example, ch. VIII of Brillouin (1938).
assumption of geometrical optics, the original scalar wave equation can be approximated by the eikonal equation for the unknown function \( L(r) \), called the optical path length. Surfaces of constant \( L \), determined by the solution of the eikonal equation, are surfaces of constant optical phase and thus define the wavefronts. The ray trajectories are everywhere perpendicular to the wavefronts because they are also determined by the eikonal equation. In this context, it is well founded to interpret the Poynting vector of electrodynamics as a ray vector.

1.4 Radiometric Concepts

Radiometry deals with the measurement of the production and propagation of observable electromagnetic fields. As thoroughly discussed in chapter II of Preisendorfer (1965), the radiometric concepts, based on the single physical idea of radiant flux, can be defined by means of the idealized results of real experiments. This is an operational approach, whose essential outcomes we are going to summarize here in the language of the mathematical theory of measure. We recall that the radiant flux is defined as the amount of radiant energy flowing per unit time across a given surface. Following Preisendorfer, the former can be operationally defined in terms of the readings of a standard meter, which senses and records the radiant flux within a given set \( F \) of frequencies incident on a collecting surface \( A \) from a set \( D \) of directions. We will denote by \( \Phi(t; F, A, D) \) the reading at time \( t \) of the meter specified by the foregoing parameters.

1.4.1 Monochromatic Radiant Flux

The range of frequencies recorded by the radiant flux meter can be selected by means of a suitable filter. If the meter is adjusted so that it records simultaneously the frequencies in two disjoint sets \( F_1 \) and \( F_2 \) (i.e., \( F_1 \cap F_2 = \emptyset \)) in a first experiment and successively the separate responses to \( F_1 \) and \( F_2 \) under the same conditions, two far-reaching properties of the function can be formalized in mathematical terms. Firstly, it turns out that

\[
\Phi(t; F_1, A, D) + \Phi(t; F_2, A, D) = \Phi(t; F_1 \cup F_2, A, D)
\]

(1.1)

This is the statement of the linear additivity of \( \Phi \) in frequency. Then, if we denote by \( m(F) \) the measure\(^5\) of the set \( F \), it holds that \( \Phi = 0 \) when \( m(F) = 0 \). This second property of \( \Phi \) expresses its absolute continuity with respect to frequency.

To produce a finite amount of radiant energy, the frequency band must also be finite. We can, however, define a monochromatic radiant flux \( \Phi_\nu \) by taking the limit

\[
\Phi_\nu(t; A, D) = \lim_{\Delta \nu \to 0} \frac{\Phi(t; A, D, \Delta \nu)}{\Delta \nu}
\]

(1.2)

where \( \Delta \nu \equiv m(F) \). Because of the preceding properties, the existence of this limit is a reasonable assumption that can be rigorously proved if we recur to theory of measure. By definition, \( \Phi_\nu \) is the partial derivative of \( \Phi \) with respect to \( \nu \). Hence monochromatic implies per unit frequency range.

1.4.2 Geometrical Properties of the Radiant Flux

Again by means of hypothetical experiments with a radiant flux meter,\(^6\) based however on real ones, the following geometrical properties of \( \Phi_\nu \) can be deduced:

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5 The theory of measure assigns a real number to any subset of a set, i.e., its measure to be interpreted as its size.

6 Fully described in ch. II, sec. 9 of Preisendorfer (1965).
Directional additivity. For any pair of almost disjoint sets such that \( D_1 \cup D_2 = D \), from the results of three successive readings \( \Phi_\nu(t; A, D_1) \), \( \Phi_\nu(t; A, D_2) \) and \( \Phi_\nu(t; A, D) \), it holds that
\[
\Phi_\nu(t; A, D_1) + \Phi_\nu(t; A, D_2) = \Phi_\nu(t; A, D_1 \cup D_2),
\]
provided that the irradiation conditions are the same at the time of the three separate measurements.

Absolute directional continuity. If \( m(D) = 0 \), then \( \Phi_\nu = 0 \).

The properties of surface additivity (iii) and absolute surface continuity (iv) are deduced likewise.

Properties (i) through (iv) characterize the monochromatic radiant flux and constitute the essence of the phenomenological foundation of radiative transfer. They justify the treatment of radiative transfer as a process essentially linear in character. In particular, different rays are necessarily independent. While this excludes interference among the rays, it brings about at the same time an intrinsic difficulty in view of the representation of the radiation field: in order to specify completely its state at any point, a single quantity (either scalar or vectorial) is not enough. It shall be necessary to take into account all the rays – virtually an infinite set – passing through the point in question. It is worthwhile to remark that properties (i) and (ii) show where the macroscopic representation to be introduced in the next section contrasts with the electrodynamic one: diffraction of electromagnetic waves yields counterexamples of (i); interference disproves (ii).

1.5 Macroscopic Picture: The Specific Intensity of the Radiation Field

The energy carried on along a given direction is the fundamental physical observable in radiative transfer. We are going to introduce now a local and directional macroscopic quantity, namely the specific intensity of the radiation field, which makes possible a description of both the local properties of the radiation field and its propagation through a medium, suitable for the mathematical treatment of radiative transfer. Such a macroscopic representation is consistent with the corpuscular picture of the flow of photons, as will be shown at the end of Section 1.6.

1.5.1 Operational Definition of the Specific Intensity

We measure the amount of radiant energy \( \Delta E_\nu(n) \) that flows during a time interval \( \Delta t \) through an oriented surface \( k \Delta \sigma \) around a point \( P_1 \) individuated by its position vector \( \mathbf{r} \), inside the solid angle \( \Delta \Omega \) around the direction of propagation \( \mathbf{n} \) and vertex at \( P_1 \), with frequency in the range \( (\nu, \nu + \Delta \nu) \). We must choose the surface element \( k \Delta \sigma \) such that at each of its points \( P' \) the value of the radiation field can be considered constant. In such a way the amount of energy that enters into any solid angle \( \Delta \Omega \) with vertex at \( P' \) will be the same. Analogously, the energy spectral distribution will not vary noticeably within the frequency band \( \Delta \nu \). (The geometrical layout is sketched in Figure 1.1.)

According to the experimental laws of radiometry, the extensive quantity \( \Delta E_\nu(n) \) results proportional to each single element of the process of measurement, that is,
\[
\Delta E_\nu(n) \propto \mathbf{n} \cdot k \Delta \sigma \Delta \Omega \Delta \nu \Delta t.
\]

That is, \( D_1 \) and \( D_2 \) have in common only a set of directions of zero measure.
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Figure 1.1. The geometrical elements of the measure of $\Delta E_{\nu}(\mathbf{n})$. During the time interval $\Delta t$, the photons that flow through $\mathbf{k} \sigma$ fill the volume $\Delta V = (\mathbf{n} \cdot \mathbf{k}) \Delta \sigma$ $\sigma$ $\Delta t$. By hypothesis, the properties of the radiation field are the same at each point of the surface $\mathbf{k} \Delta \sigma$, hence within each of the three solid angles drawn.

As a consequence of the foregoing radiometric properties and the absolute continuity in frequency, the limit

$$\lim_{\Delta \sigma \Delta \Omega \Delta \nu \Delta t \to 0} (\mathbf{n} \cdot \mathbf{k})^{-1} \Delta E_{\nu}(\mathbf{n}) \equiv I(r, t; \mathbf{n}, \nu)$$

exists and takes on a finite value. This coefficient of proportionality between the measured value of the physical magnitude and the product of the values of the geometrical, spectral and time elements of the process of measurement is by definition the specific intensity of the radiation field. In such a way we have defined operationally the amount of specific energy carried on along a ray.

The dimension of $I$ is $(M L^2 T^{-2}) \cdot L^{-2} \cdot T \cdot T^{-1} = MT^{-2}$, as it follows from (1.5). That is to say, $I$ has the dimension of an energy flux per unit time and unit frequency band; in other words, a monochromatic power flux. In the centimetre-gram-second (c.g.s.) system, the units of $I$ are $\text{erg cm}^{-2} \text{st}^{-1} \text{hz}^{-1} \text{s}^{-1}$. Dimensional analysis shows that the specific intensity has to be identified with the radiance, as defined in radiometry.

For further use, it will be convenient to write explicitly the infinitesimal amount of radiant energy $dE_{\nu}(\mathbf{n})$ as a function of the specific intensity $I(r, t; \mathbf{n}, \nu)$ and the geometrical, spectral and temporal parameters involved, that is,

$$dE_{\nu}(\mathbf{n}) = I(r, t; \mathbf{n}, \nu) \, \mathbf{n} \cdot \mathbf{k} \, d\sigma \, d\Omega \, d\nu \, dt.$$  

(1.6)

1.5.2 Moments of the Specific Intensity

The straight average of the specific intensity over all solid angles defines the mean intensity of the radiation field, namely

$$J(r, t; \nu) = \frac{1}{4\pi} \int I(r, t; \mathbf{n}, \nu) \, d\Omega.$$  

(1.7)

The mean intensity $J$ has the same dimension of the specific intensity $I$, but its units in the c.g.s. system are $\text{erg cm}^{-2} \text{hz}^{-1} \text{s}^{-1}$.

The mean intensity, defined in this way, is the zero-order moment with respect to $\mathbf{n}$ of the specific intensity. Two successive moments are defined likewise:
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1. The first-order moment

\[ F_\nu (r,t) \equiv \oint I(r,t : n, \nu) n \, d\Omega. \]  

(1.8)

2. The second-order moment:

\[ T_\nu \equiv \frac{1}{c} \oint I(r,t; n, \nu) nn \, d\Omega. \]  

(1.9)

The first-order moment is a vector to be identified with the flux of radiation; the second-order moment is a tensor that we will identify in the following with the radiation pressure tensor. (See Sections 1.13.2 and 1.13.3.)

1.5.3 **Energy Density of the Radiation Field**

Let us compute now the specific energy density of the radiation field in the frequency range \((\nu, \nu + d\nu)\). The photons that carry on the amount of energy \(dE_\nu(n)\) specified by (1.6) fill during the time interval \(dt\) the volume \(dV = n \cdot k \, d\sigma \, dt\). (See Figure 1.1.)

We define the specific energy density as

\[ U(r,t; n, \nu) \, d\Omega \, d\nu \equiv \frac{dE_\nu(n)}{d\nu} = \frac{1}{c} I(r,t; n, \nu) \, d\Omega \, d\nu. \]  

(1.10)

By integration of (1.10) over all the solid angles, it follow

\[ \frac{d\nu}{c} \oint I(r,t; n, \nu) \, d\Omega = \frac{4\pi}{c} J(r,t; \nu) \, d\nu. \]  

(1.11)

Hence we may define the monochromatic energy density (i.e., the energy density per unit frequency band) as

\[ u_\nu(r,t) \equiv \frac{4\pi}{c} J_\nu(r,t). \]  

(1.12)

The dimension of \(u_\nu\) is \((ML^2 T^{-2}) L^{-3} T^{-1}\) and its c.g.s. units are \(erg \ cm^{-3} \ hz^{-1}\). The integration of \(u_\nu\) over the whole frequency range yields the bolometric energy density \(u(r,t)\) that accounts for the localization of energy within the radiation field.

1.6 **Microscopic Picture: The Corpuscular Model**

It is needless to recall that, for a physical system constituted by very, very many particles, it is unfeasible to describe their individual properties. The only way out is a statistical approach, based on the average behaviour of a large number of them. In order to study the flows of energy and momentum inside a given system, information is required about the spatial and velocity distribution of the constituting particles. The simplest function that contains the necessary information is a distribution function defined on the phase space so that \(f(r, v, t) \, d^3r \, d^3v\) is the expected number of particles at time \(t\) inside the infinitesimal volume \(d^3r\) around \(r\) with velocities in the infinitesimal volume \(d^3v\) around \(v\). The usual assumption is that \(d^3r\) is small compared with the spatial variation of any macroscopic propriety of the system, but large enough to contain a statistically significant number of particles.

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8 We have rather to employ here the dyadic notation, introduced by J. W. Gibbs in 1884. Although relatively obsolete nowadays, it is employed in continuum mechanics and electromagnetism.
The corpuscular nature of light suggests a picture in terms of a *flow of particles*; photons with energy $h\nu$ that carry on a momentum $p = (h\nu/c)\mathbf{n}$ in their flight along a given direction specified by the unit vector $\mathbf{n}$. We are going to introduce two photon distribution functions, namely $f(r, t; \mathbf{n}, \nu)$ and $F(r, p, t)$. The former, characterized by the pair of parameters $(\mathbf{n}, \nu)$ is consistent with the macroscopic description of the transport of radiant energy in terms of the specific intensity of the radiation field. The latter has the standard form of a distribution function, whose variables are $r$, $p$ and $t$.

### 1.6.1 First Photon Distribution Function

For the sake of a direct comparison with the specific intensity of the radiation field $I(r, t; \mathbf{n}, \nu)$, we will define a distribution function $f$ such that $f(r, t; \mathbf{n}, \nu) \ d\Omega d\nu$ yields the number of photons per unit volume, at location $r$ and time $t$ in the range $(\nu, \nu + d\nu)$ that propagate along the direction $\mathbf{n}$ with speed $c$ within the solid angle $d\Omega$ around $\mathbf{n}$. The dimension of $f$ is $L^{-3} T$.

The specific photons $(\mathbf{n}, \nu)$ crossing $k \cdot d\sigma$ during the time interval $dt$ fill a volume $dV = \mathbf{n} \cdot \mathbf{k} \ d\sigma \ c \ dt$, hence $f(r, t; \mathbf{n}, \nu) = \mathbf{n} \cdot \mathbf{k} \ d\sigma \ c \ dt \ d\Omega d\nu$ will be their number. The corresponding energy is

$$dE_r(n = h\nu c f(r, t; \mathbf{n}, \nu) \mathbf{n} \cdot \mathbf{k} \ d\sigma \ d\Omega d\nu dt.$$  \hspace{1cm} (1.13)

By direct comparison with (1.6), it follows that

$$I(r, t; \mathbf{n}, \nu) = h\nu c f(r, t; \mathbf{n}, \nu).$$ \hspace{1cm} (1.14)

Equation (1.14) gives the quantitative link between the macroscopic description and the corpuscular picture of the radiation field.

### 1.6.2 Second Photon Distribution Function

In order to formulate the radiative transfer equation as a transport equation like in the kinetic theory of gases, we will make use of a second photon distribution function, defined so that $F(r, p, t) \ dr \ dp$ gives the number of photons per unit volume at point $r$ and time $t$ with momentum in the range $(p, p + dp)$. The dimension of $F$ is $M^{-3} L^{-6} T^3$. The momentum $\mathbf{p}$ is linked to the frequency $\nu$ and the direction of propagation $\mathbf{n}$ of the photons according to the relation

$$\mathbf{p} = \frac{h\nu}{c}\mathbf{n}. \hspace{1cm} (1.15)$$

That is to say $\mathbf{p} = \mathbf{p}(\mathbf{n}, \nu)$ can be expressed as a function of the pair of parameters $\mathbf{n} = \mathbf{n}(\vartheta, \varphi)$ and $\nu$. The infinitesimal volume in the $R^3$ momentum space is

$$d^3 p = dp_x \ dp_y \ dp_z = p^2 \sin \vartheta \ dp \ d\vartheta \ d\varphi. \hspace{1cm} (1.16)$$

From (1.15), it follows that $p = (h\nu/c)$ and $dp = (h/c) d\nu$, hence

$$d^3 p = \frac{h^3\nu^2}{c^3} \sin \vartheta \ d\vartheta \ d\varphi \ d\nu. \hspace{1cm} (1.17)$$

In the $R^3$ direction $\times$ frequency parameter space, the infinitesimal volume is given by

$$d\Omega \ d\nu = \sin \vartheta \ d\vartheta \ d\varphi \ d\nu, \hspace{1cm} (1.18)$$
where \( d\Omega \) is the solid angle around \( n \). According to the previous definitions it follows that

\[
f(r, t; n, \nu) \ d^3r \ d\Omega \ d\nu = F(r, p, t) \ d^3r \ d^3p.
\]

(1.19)

Hence, taking into account (1.17) and (1.19), it follows straightforwardly that

\[
f(r, t; n, \nu) = \frac{\hbar \nu^2}{c^2} F(r, p, t).
\]

(1.20)

Equations (1.14) and (1.20) proves that the specific intensity \( I(r, t; n, \nu) \) is proportional to the photon distribution function \( F(r, p, t) \). This result shows the correspondence between the macroscopic description of the radiation field and the corpuscular picture.

1.7 The Radiative Transfer Equation as a Kinetic Equation for Photons

In the kinetic theory of gases the distribution function \( F(r, p, t) \) is assumed to vary with time because the particles constantly enter and leave a given volume of the phase space. The change is due to the streaming of the particles and binary collisions among them. This process can be expressed in words as

Total rate of change = Source terms – Sink terms.

The translation of this statement into mathematical language leads to the Boltzmann transport equation:

\[
\frac{d}{dt} F(r, p, t) = \left( \frac{\partial}{\partial t} + v \cdot \nabla_r + \frac{F_{\text{ext}}}{m} \cdot \nabla_v \right) F(r, p, t)
\]

(1.21)

where \( \nabla_r \) and \( \nabla_v \) denote the gradient with respect to \( r \) and \( v \), respectively, and \( F_{\text{ext}} \) the external force acting on a particle of mass \( m \). The right-hand side (RHS) is the Boltzmann collisional operator, which accounts for the kinetic balance between gains (sources) and losses (sinks).

It follows from (1.14) and (1.20) that the specific intensity \( I(r, t; n, \nu) \) is proportional to the photon distribution function \( F(r, p, t) \). Because there are not external forces acting on the photons and \( v = cn \), we can write the transport equation for the distribution function \( I(r, t; n, \nu) \) as

\[
\frac{1}{c} \frac{\partial I(r, t; n, \nu)}{\partial t} + n \cdot \nabla I(r, t; n, \nu) = \frac{1}{c} \left[ \frac{\delta I_n}{\delta t} \right]^+ - \frac{1}{c} \left[ \frac{\delta I_n}{\delta t} \right]^-.
\]

(1.22)

The dimension of the terms in (1.22) is \((ML^2T^{-2}) \cdot L^{-3}\), that is, an energy density.

Equation (1.22) is the mathematical formulation of a directional problem: the transport of a local and scalar quantity along a given direction \( n \). As \( \delta l = c\delta t \) is a path length along \( n \), the left-hand side (LHS) of (1.22) gives the variation per unit length of the specific intensity that propagates in the direction \( n \). This variation must be equal to the difference

\[9\] The derivation of the Boltzmann equation is nicely sketched in section 2-2 of Mihalas (1978). Among the text books on kinetic theory, Huang (1963) covers in full details the physics of transport processes.