CONCEPTIONS OF SET AND THE FOUNDATIONS OF MATHEMATICS

Sets are central to mathematics and its foundations, but what are they? In this book Luca Incurvati provides a detailed examination of all the major conceptions of set and discusses their virtues and shortcomings, as well as introducing the fundamentals of the alternative set theories with which these conceptions are associated. He shows that the conceptual landscape includes not only the naïve and iterative conceptions but also the limitation of size conception, the definite conception, the stratified conception and the graph conception. In addition, he presents a novel, minimalist account of the iterative conception which does not require the existence of a relation of metaphysical dependence between a set and its members. His book will be of interest to researchers and advanced students in logic and the philosophy of mathematics.

LUCA INCURVATI is Assistant Professor of Philosophy at the University of Amsterdam.

CONCEPTIONS OF SET AND THE FOUNDATIONS OF MATHEMATICS

LUCA INCURVATI
University of Amsterdam

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India

79 Anson Road, #06–04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781108497824
DOI: 10.1017/9781108596961

First published 2020

Printed in the United Kingdom by TJ International Ltd, Padstow Cornwall.

A catalogue record for this publication is available from the British Library.

ISBN 978-1-108-49782-4 Hardback

Contents

Figures

Tables

Preface

I took my first set theory course as a philosophy student at the Department of Mathematics of the University of Rome La Sapienza. It was after taking that course, taught by Professor Claudio Bernardi, that I decided to dedicate more time and energy to the philosophy of set theory if I could.

The course was entitled 'Foundations of Mathematics' and had one striking feature: it was taught from the ZFC axioms, with almost no mention of naïve set theory and the set-theoretic paradoxes. When I went up to Cambridge in 2005, I began to read some of the standard textbooks on set theory, such as Devlin 1993 and Jech 2003. Naïve set theory was now introduced and the paradoxes were given their due. But after this, ZFC was presented, and one was left with the impression that all roads from naïve set theory lead to the cumulative hierarchy.

This book is an attempt to reverse this trend. Perhaps, in the end, we should stick with the iterative conception of set. Indeed, one can read the book as an extended argument to this effect. But the path from the paradoxes to ZFC and cognate systems is much more tortuous than tradition has made it out to be.

The book describes and assesses a number of *conceptions of set*. Being a book on the *philosophy* of set theory, the focus is very much on the philosophical underpinnings of these conceptions. But history is important, in philosophy as in life, and I have provided some historical background when I deemed it useful.

Being a book on the philosophy of *set theory*, the book deals with technical material. I have tried to make the material as accessible as possible and have included some sections introducing some of the basic notions used, namely Section 1.1 and the appendices to Chapter 1. More advanced notions from set theory are introduced in the body of the text or in footnotes.

The book deals with some theories that are likely to be less known to philosophical audiences (e.g. the non-well-founded set theories of Chapter 7) and mathematical audiences (e.g. certain set theories based on the naïve conception of set, presented in Chapter 4). The book can also serve as a friendly introduction to these set theories, whose central formal aspects are described in the relevant chapters.

Parts of the book are based on material which has previously appeared in print. In particular, Chapters 2 and 7 draw, respectively, on 'How to be a minimalist about sets' (*Philosophical Studies*, 2012) and 'The graph conception of set' (*Journal of Philosophical Logic*, 2014). I am grateful to Springer for permission to use material from these two articles. In addition, parts of Section 3.6 were originally published as section 2 of 'Maximality principles in set theory' (*Philosophia Mathematica*, 2017), and I would like to thank Oxford University Press for permission here.

* * *

A summary of the content of the individual chapters will be helpful. The first chapter explains what conceptions of set are, what they do, and what they are for. It begins by introducing the concept of set and making explicit certain assumptions about concepts. Following a tradition going back to Frege, concepts are taken to be equipped with criteria of application and, in some cases, criteria of identity. It is pointed out that a concept need not settle all questions concerning which objects it applies to and under what conditions those objects are identical. This observation is used to characterize conceptions: a conception settles more questions of this kind than the corresponding concept does. On this account, a conception *sharpens* the corresponding concept. This view is contrasted with two other possible views on what conceptions do: one according to which they provide an *analysis* of what belongs to a concept, and one according to which they are possible *replacements* for a given concept. The chapter concludes with a discussion of the uses of conceptions of set. Along the way, the *naïve conception of set*, which holds that every condition determines a set, is introduced and a diagnosis of the set-theoretic paradoxes is offered. According to this diagnosis, the naïve conception leads to paradox because it requires the concept of set to have two features – *indefinite extensibility* and *universality* – which are jointly inconsistent.

The second chapter introduces the *iterative conception*, according to which every set appears at one level or another of the mathematical structure known as the *cumulative hierarchy*, as well as theories based on the conception. The chapter presents various accounts of the iterative conception: the *constructivist* account, the *dependency* account and my own *minimalist* account. It is argued that the minimalist account is to be preferred to the others. A method – which I call *inference to the best conception* – is then described to defend the correctness of the iterative conception so understood. This method requires one to show that the iterative conception fares better than other conceptions with respect to a number of desiderata on conceptions of set. This provides additional motivation for exploring alternative conceptions of set in the remainder of the book.

The main purpose of the third chapter is to defend the iterative conception against three objections. The first objection, which I call the *missing explanation objection*, is that if the iterative conception is correct, one cannot explain the intuitive appeal of the Naïve Comprehension Schema. The chapter provides plausible explanations of this fact which are compatible with the correctness of the conception. The second objection, which I call the *circularity objection*, is that the iterative conception presupposes the notion of an ordinal, and since ordinals are treated in set theory like certain kinds of sets, this means that the conception is vitiated by circularity. The chapter shows that this objection can be defeated by constructing ordinals using a trick that goes back to Tarski and Scott or dispensing with the notion of well-ordering altogether in the formulation of the conception. The third objection, which I call the *no semantics objection*, is that the iterative conception prevents us from giving a semantics for set theory, since according to the conception there is no universal set but we seem to need such a set to serve as the domain of quantification when giving a semantics for set theory. The chapter defends the approach that this problem can be overcome by doing semantics in a higher-order language. The chapter concludes by discussing the status of the Axiom of Replacement on the iterative conception.

The fourth chapter discusses the naïve conception of set and criticizes attempts to rehabilitate it by modifying the logic of set theory. The focus is on the proposal that the Naïve Comprehension Schema – which formally captures the thesis that every condition determines a set – is to be saved by adopting a paraconsistent logic. Three strategies for doing so are distinguished: the *material* strategy, the *relevant* strategy and the *model-theoretic strategy*. It is shown that these strategies lead to set theories that are either too weak or ad hoc or give up on the idea that sets are genuinely extensional entities.

Chapters 5 and 6 consider attempts to modify the naïve conception of set by restricting it in an appropriate manner. The idea is that the core thought of the naïve conception – that there is an intimate connection between sets and properties – can be preserved as long as we build into the conception the idea that certain properties are pathological and, for this reason, do not determine a set. The fifth chapter first uses a result from Incurvati and Murzi 2017 to show that restricting attention to those properties that do not give rise to inconsistency will not do. It then focuses on the *limitation of size conception of set*, according to which the pathological properties are those that apply to too many things. Various versions of the doctrine are distinguished. The chapter also discusses what it calls the *definite conception*, according to which the pathological properties are the indefinitely extensible ones. It is argued that the limitation of size fails to provide a complete explanation of the set-theoretic paradoxes. The definite conception faces the same problem and, in addition, it is unclear whether it has the resources to develop a reasonable amount of set theory.

The sixth chapter considers attempts to modify the Naïve Comprehension Schema according to syntactic notions and in particular the technical notion of stratification used in Quine's New Foundations. This and related theories (such as New Foundations with *Urelemente*) are examined and their philosophical underpinnings discussed and assessed. It is argued that, contra what has often been suggested in the literature, one can describe a conception of set – the *stratified conception* – which is well motivated whilst incorporating the idea that the pathological properties are those whose syntactic expression does not satisfy the stratification requirement. However, it is argued that this conception is best seen as a conception of objectified properties rather than as a conception of sets as combinatorial collections. Understanding New Foundations and cognate systems as theories of objectified properties leads to a further development of the diagnosis of the set-theoretic paradoxes offered in the first chapter and allows one to deal with some of the standard objections to set theories based on stratification.

The seventh chapter presents and discusses the *graph conception of set*. According to the graph conception, sets are things depicted by graphs of a certain sort. The chapter begins by presenting four set theories, due to Aczel, which are formulated by using the notion of a graph. The graph conception is then introduced, and a historical excursion into forerunners of the conception is also given. The chapter continues by clarifying the relationship between the conception and the four theories described by Aczel. It concludes by discussing four objections to the graph conception: the objection that set theories based on graphs do not introduce new isomorphism types; the objection that the graph conception does not provide us with an intuitive model for the set theory it sanctions; the objection that the graph conception cannot naturally allow for *Urelemente*; and the objection that a set theory based on the graph conception cannot provide an autonomous foundation for mathematics. It is argued that whilst the first two objections fail, the remaining two retain their force.

The eighth and last chapter offers some concluding remarks. It provides an overview of some of the central features of the conceptions of set encountered in the book. It then discusses to what extent the findings of the book are compatible with some form of pluralism about conceptions.

* * *

This book is the result of thinking that spans more than a decade. As my ideas evolved, I was fortunate enough to be able to present them to a number of audiences who provided useful feedback. These include the Moral Sciences Club in

Cambridge, the Foundations of Mathematics Seminar at the University of Paris 7-Diderot, the seminar of the project *Plurals, Predicates and Paradox* in London, the Third Paris-Nancy Philosophy of Mathematics Workshop, the Philosophy of Mathematics Seminar in Oxford, the Faculty of Philosophy of the Università Vita-Salute San Raffaele, the 88th Joint Session of the Aristotelian Society and the Mind Association, the Department of Philosophy of the University of Amsterdam and the seminar of the Logic Group of the University of Hamburg.

The following set of *Urelemente* deserves special mention: {Arianna Betti, Robert Black, Francesca Boccuni, Tim Button, Mic Detlefsen, Salvatore Florio, Volker Halbach, Leon Horsten, Dan Isaacson, Hannes Leitgeb, Øystein Linnebo, Benedikt Löwe, Julien Murzi, Alex Oliver, Alexander Paseau, Michael Potter, Ian Rumfitt, Stewart Shapiro, Rob Trueman, Sean Walsh, Nathan Wildman, Tim Williamson}. No doubt this set is a proper subset of the set of *Urelemente* to whom I am indebted. I am also grateful to Hilary Gaskin at Cambridge University Press for her help and support throughout this project and to two referees, whose detailed and insightful comments on earlier versions of this material led to its final shape. Finally, I would like to thank my parents and Sarah for their constant and loving support. This book is dedicated to the memory of my late grandparents.

Cambridge University Press
978-1-108-49782-4 — Conceptions of Set and the Foundations of Mathematics
Luca Incurvati
Frontmatter
More Information