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978-1-108-49782-4 — Conceptions of Set and the Foundations of Mathematics
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Concepts and Conceptions

> Class *presented itself as a vague notion, or, specifically, a mixture of notions.*
>
> *Georg Kreisel (1967: 82)*

This book, as its title indicates, is about *conceptions of set*. Conceptions of set are advocated by mathematicians and philosophers with different, but often related purposes in mind. In this chapter, we offer a preliminary discussion of what we can take conceptions of set to be, what they are supposed to do, and the goals they are supposed to achieve. In so doing, we will provide a map of the wider territory within which the debate about conceptions of set takes place. Along the way, we shall briefly encounter our first conception of set, namely the *naïve conception*.

1.1 Theories

Before we get started, however, it will be helpful to remind the reader of a few notions we will use throughout. This will also help to fix terminology.

Conceptions of set, as we shall see, are often invoked to justify *theories* of set. At the most general level, a *theory* T in a given language consists of some sentences of that language. But the theories that conceptions of set are invoked to justify are typically *axiomatic formal theories*.

Axiomatic formal theories are *axiomatic* because they have (non-logical) *axioms*. That is to say, some of their sentences are singled out as those sentences which are not in need of proof and from which the theorems are derived. These derivations are carried out using a particular *logic*. For the most part, we will be concerned with *classical first-order logic*, but we will have occasion to look at theories that are cast in logics which extend this logic or deviate from it. For instance, we

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will look at theories that are cast in classical *second-order logic*, which we shall distinguish by adding a '2' as a subscript to their name, and at theories that are cast in *paraconsistent* logic. Relevant details will be given in due course.

Axiomatic formal theories are *formal* because the language they are cast in is a *formalized* language. A formalized language $\mathcal{L}$ is such that there are precise rules for determining which sequences of symbols constitute a *well-formed formula* of $\mathcal{L}$, and a sentence of $\mathcal{L}$ is simply a well-formed formula with no free variables. The symbols of a formalized language can be divided into those that make up its *logical vocabulary* – in our case, variables, bracketing devices and symbols for connectives, quantifiers and identity – and those that make up its *non-logical vocabulary* – for instance, individual constants and predicates.

Sometimes, when talking of a formalized language one is only referring to the syntax of that language. Other times, however, one is referring to the language as endowed with meaning, and, in the context of axiomatic formal theories, we will be using the term 'language' in both ways. In the latter case, the idea is that the language has a particular *interpretation*. Standardly, an *interpretation* for a given language $\mathcal{L}$ is an ordered pair $\mathfrak{A} = \langle \mathcal{D}, \mathcal{I} \rangle$ where $\mathcal{D}$ is the domain of interpretation – usually thought of as a non-empty set – and $\mathcal{I}$ is an interpretation function, i.e. a function satisfying the following conditions:

- $\mathcal{I}(c)$ is an element of $\mathcal{D}$ when c is a name of $\mathcal{L}$;
- $\mathcal{I}(E)$ is a set of n-tuples of $\mathcal{D}$-elements when E is an n-place predicate.

A *model* of a theory is then an interpretation which makes every sentence of the theory true.[1] Note that this means that, officially, an interpretation is a set-theoretic entity. In Section 3.4, we will discuss an approach to model theory which is different in this respect.

Finally, we will follow custom and will sometimes use T to refer solely to the axioms of T rather than to the deductive closure of these axioms – that is, the theorems that can be derived from T's axioms in T's logic. The context will make it clear when this is the case.

1.2 The Concept of Set

Unsurprisingly, conceptions of set are best explained by reference to the *concept* of set. On the face of it, English speakers possess this concept: if I say that the set of books on my table has two elements, you understand what I am saying. Moreover, English speakers with some knowledge of basic, secondary school mathematics

[1] For more on the notions of model and truth in an interpretation, see Section 3.3.

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seem to understand sentences such as 'the set of natural numbers is infinite' and 'every set of real numbers which has an upper bound has a least upper bound'. Thus, not only do speakers seem to possess the concept of set as it occurs in everyday parlance, but they also seem to have the concept of set as it occurs in elementary mathematics.

Things, however, are not as simple as they may seem at first sight. Quite often, when people talk of a set of things, what they say can easily be recast without any mention of sets. So, for instance, when I said that the set of books on my table has two elements, it would have been prima facie legitimate for someone to take me simply to be saying that there are two books on my table. In other words, 'set of things' is often used, in ordinary parlance, as synonymous for what philosophers call a *plurality* of these things. A plurality, in this sense, is not something over and above the things comprising it: it is just them.

Other terms used by philosophers and mathematicians to talk about the same sort of entities are 'totality', 'multiplicity', 'multitude' and, sometimes, 'class'. With regard to the latter, Paul Finsler, as early as 1926, makes the point as follows:

> It would surely be inconvenient if one always had to speak of many things in the plural; it is much more convenient to use the singular and speak of them as a *class*. [. . .] A class of things is understood as being the things themselves, while the set which contains them as its elements is a single thing, in general distinct from the things comprising it. [. . .] Thus a set is a genuine, individual entity. By contrast, a class is singular only by virtue of linguistic usage; in actuality, it almost always signifies a plurality. (Finsler 1926: 106)

Depending on what stance one takes on whether there are pluralities consisting of just one object or of no object at all, pluralities may be regarded as either a special case of, or identified with, what Bertrand Russell (1903) calls *classes as many*. The notion of a class as many is best understood by distinguishing between three cases in which we seem to have the class as many of the things falling under a concept C.

First, there is the case in which nothing falls under C. In this case, there is no class as many of the things falling under C. As Russell (1903: §69) puts it, 'there is no such thing as the null class, though there are null class-concepts'. Second, there is the case in which exactly one thing falls under C. Letting $[x : x \text{ is } \Phi]$ denote the concept under which the things that are Φ fall (or the property had by the Φ-things), an example is provided by the concept $[x : x$ is a current Chancellor of Germany$]$. In this case, the class as many of the things falling under C is the thing itself – Angela Merkel, in the case we are considering (as of 2019). Using Russell's (1903: §69) words: 'a class having only one term is to be identified [. . .] with that one term'. Third, there is the case in which more than one objects falls under the concept whose class as many we are considering. In that case, the class as many is

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just the plurality of the things falling under that concept. And, as Russell observes, '[i]n such cases, though terms may be said to belong to the class, the class must not be treated as itself a single logical subject' (Russell 1903: §70).

In general, then, a class as many is to be identified with the things falling under a certain concept, be they none, one or many. Bearing this fact in mind, we can now begin to say a bit more about the concept we are focusing on in this book, the concept of set. For one central feature of sets is that they are single objects, they are *unities*, even in the case in which there is more than one object comprising them. A set is a 'genuine, individual entity', as Finsler put it in the quote above – it is a *class as one*, to use Russell's terminology (1903: §74). Thus, in at least certain cases, some things yy will form a set, which is a single entity. Such a set will be the set a of all x such that x is one of the yy, which we write $\{x | x \text{ is one of the } yy\}$. We say that each b among the yy is a *member* of (or *element* of) a, which we write $b \in a$. Similarly, we write $b \notin a$ to indicate that b is *not* a member of a.

Let us now return to the example of the set of the natural numbers. The idea is that the plurality of my hands, which for the occasion we may call Left and Right, just is my hands – just is Left and Right. The set {Left, Right}, on the other hand, is a single object, whose members are Left and Right. Or consider again our earlier example about the set of natural numbers. Although when we talk about this set there is a reading according to which we are talking about *the natural numbers themselves*, we will be concerned with the reading according to which we are talking about *the set*, understood as a genuine entity over and above the natural numbers.

This central aspect of the concept of set was emphasized time and again by the founder of set theory, Georg Cantor. In an oft-quoted passage he writes:

> By a 'set' we understand every collection to a whole M of definite, well-differentiated objects m of our intuition or our thought. (Cantor 1895: 282)

Similar remarks appear elsewhere. For instance, two years earlier Cantor had written:

> By a 'manifold' or 'set' I understand in general any many [*Viele*] which can be thought of as one [*Eines*], that is, every totality of definite elements which can be united to a whole through a law. (Cantor 1883: 204, fn. 1)

And in a 1899 letter to Dedekind he tells us that

> [i]f [. . .] the totality of the elements of a multiplicity can be thought of without contradiction as 'being together', so that they can be gathered together into '*one* thing', I call it a *consistent multiplicity* or a 'set'. (Cantor 1899: 114)

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The last characterization of the notion of set appeals to consistency, and we shall return to this aspect of Cantor's thought in Chapter 5. The second quotation appeals to the idea that it is through a *law* that a plurality becomes a unity – an idea which is absent in the other quotes. Finally, in all passages some mention is made of the role that our *thought* has in determining whether a certain plurality forms a set, which is something that – as we shall see in the next chapter – raises a number of issues. But the feature of sets which is mentioned in all passages and which, following Cantor, is now standardly taken to be part of the concept of set and distinguishes from the concept of a plurality is that a set is *one thing*.

Cantor is also explicit that, as we emphasized, a set is an entity over and above its members:

> This set [the set of all natural numbers] is a thing in itself and constitutes, completely apart from the natural sequence of numbers belonging to it, a firm in all its parts, determinate quantum, an *apōrisménon*. (Cantor 1887–88: 401, trans. in Jané 1995: 393)

We are now in a position to lay down one central feature of the concept of set:

> **Unity of Sets.** A set is a *unity*, i.e. a single object, over and above its members.

In particular, a set is a single object bearing a certain distinguished relation – the containment relation (the converse of the membership relation) – to the objects a class as many *aa* comprises. Note, however, that we are making no presumption that to each class as many *aa* there corresponds the set of all the *aa*.

The fact that sets are unities helps distinguish between sets and pluralities. However, it fails to distinguish between sets and what philosophers call *fusions*. A fusion of some entities, in the philosopher's sense, is the same as the *mereological sum* of those entities – that is, the sum of those entities considered as parts of the whole that consists of them.[2] Thus, a is a fusion of b, c and d if b, c and d are parts of a and every part of a shares a part with b, c or d. Hence, a dog is the fusion of the *molecules* which make it up, but it is also a fusion of the *cells* which make it up.

Now some philosophers claim that a fusion is nothing over and above its parts, and is instead *identical* to them. On this view, known as *composition as identity*, a fusion is identical to a plurality – namely the plurality of things that make it up – but is nonetheless a genuine, individual entity.[3] We could try to rephrase Unity of Sets

[2] Mereology, the study of the parthood relation, was initiated, in its contemporary form, by Leśniewski (1916). For a historical introduction, see Simons 1991.

[3] The modern debate on whether composition might be identity begins with Baxter 1988. For a contemporary overview including some recent contributions, see Cotnoir and Baxter 2014.

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Table 1.1 *Comparing pluralities, fusions and sets*

	Unity	Unique Decomposition
Plurality		✓
Fusion	✓	
Set	✓	✓

so that it requires a set to be *only* an individual entity, so that it cannot be identical to any plurality.[4] But even then, it would remain the case that most philosophers *reject* composition as identity and take instead a fusion to be a single entity different from its parts.

Thus, Unity of Sets cannot serve to distinguish sets from fusions. Instead, what distinguishes sets from fusions is that fusions need not have a *unique decomposition*: as we pointed out, a dog can be decomposed into both molecules and cells. Hence, there is more than one list of things such that they are parts of the dog and their sum *is* the dog. By contrast, sets do have a unique 'decomposition' into members: for any set a, there is one and only one list of things such that they are members of a and their union is a. Thus, whilst fusions need not uniquely decompose into parts, sets do uniquely decompose into members.

This gives rise to another central feature of the concept of set:

> **Unique Decomposition of Sets.** A set has a *unique decomposition*.

Unlike fusions, therefore, sets have a unique decomposition. But pluralities do too: given a plurality aa of objects, there is only one list of things that are among the aa and such that their union is aa.[5] The situation is summed up in Table 1.1. We have thus located two features belonging to the concept of set: the fact that sets are the result of collecting a plurality into a unity and the fact that this unity has a unique decomposition into members. These features, however, do not suffice to distinguish sets from other ways of collecting pluralities into unities. To deal with such cases, we first need to say something more general about concepts and introduce the notion of a criterion of identity.

4 Potter (2004) takes it to be part of the concept of a fusion that a fusion is nothing over and above its parts, and cites Lewis (1991) to this effect. But note that Lewis is defending a specific view of what fusions are, namely composition as identity.

5 At least, this is the received view on the matter. Ted Sider (2007) has argued that if a strong form of composition as identity holds, then pluralities do not satisfy Unique Decomposition either.

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1.3 Criteria of Application and Criteria of Identity

Thus far we have been talking in a rather unreflective manner about the concept of set. What concepts are is a controversial matter, one on which we need not take a particular stand in this book. It is typically agreed, however, that concepts are associated, one way or another, with what Michael Dummett (1981: 73ff.) calls a *criterion of application*.

Roughly speaking, a criterion of application for a concept C tells us what objects fall under C – to which objects C applies. Consider $[x : x$ is a bachelor$]$. This concept seems to be associated with the following criterion of application:

$$\text{‘bachelor’ applies to } x \text{ iff } x \text{ is unmarried and } x \text{ is a man.}^{6} \tag{1.1}$$

Thus, someone who possesses the concept of bachelor will, in normal circumstances, be willing to apply ‘bachelor’ to an object just in case they take that object to be an unmarried man. (This account of concept possession would need to be refined, but it will do for current purposes.)

Now there is a sense in which the criterion of application for a concept is *constitutive* of that concept: the meaning of ‘bachelor’ seems to be determined, at least in part, by (1.1). It is perhaps worth stressing that a criterion of application for a concept need not be that in terms of which concepts are explained: someone might accept that a concept has a criterion of application whilst explaining concepts in terms of a variety of other notions such as, e.g., mental representation or inferential role. What matters is that concepts are typically taken to be associated with criteria of application, which are partly constitutive of that concept.

But what about the concept $[x : x$ is a set$]$? If people do possess the concept of set, then this concept must be associated with a criterion of application. We shall shortly return to the question of what this criterion of application might be. For the time being, we need to notice that besides a criterion of application, certain concepts also have what Dummett calls a *criterion of identity*.

A criterion of identity specifies the conditions under which some thing x falling under a concept C is the same as another thing y, also falling under C.[7] It was Gottlob Frege who first pointed out in *Die Grundlagen* (1884: §54) that only certain concepts are associated with a criterion of identity.

Consider the concept $[x : x$ is red$]$. Someone who possesses this concept is, in the majority of cases, able to tell whether ‘red’ applies or fails to apply to a certain

[6] Here and throughout, I use ‘iff’ as an abbreviation of ‘if and only if’.

[7] Of course, each thing can only be identical to *itself*, so if that is the case it is not really *another* thing. Similarly, often people ask whether two things are identical, where clearly, if they are, they are not really *two* things. I take it to be clear enough what is meant when using locutions of this kind, and shall therefore indulge in them.

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object: she will in general be able to determine when it is correct to say 'That is red'. But there seems to be nothing in the meaning of 'red' which enables one to determine whether two red things are the same or not. Thus, $[x : x \text{ is red}]$ is associated with a criterion of application but not with a criterion of identity. This highlights a contrast with concepts such as $[x : x \text{ is a table}]$: not only is someone who possesses this concept capable of telling, in many situations, whether an object is a table or not, but they are also able to tell whether a table x is the very same table as the table y. $[x : x \text{ is a table}]$ is associated with a criterion of identity as well as a criterion of application.

The discussion so far suggests that a *criterion of identity* should specify when a thing x (falling under concept C) is identical to a thing y (also falling under C) in terms of a relation Φ holding between x and y. Formally, this gives rise to the following:

$$\forall x \forall y (K(x) \wedge K(y) \rightarrow (x = y \leftrightarrow \Phi(x, y))), \qquad \text{(CI)}$$

where '$K(x)$' expresses x *falls under* C. Note that since identity is an equivalence relation, the embedded relation Φ must be an equivalence too.

A notorious example is Donald Davidson's (1969) criterion of identity for events, according to which two events are the same just in case they have the same causes and effects:

$$\forall x \forall y (\text{Event}(x) \wedge \text{Event}(y) \rightarrow (x = y \leftrightarrow (\forall z (z \text{ is a cause of } x \leftrightarrow z \text{ is a cause of } y) \wedge \forall z (z \text{ is an effect of } x \leftrightarrow z \text{ is an effect of } y))). \qquad \text{(CIE)}$$

However, some philosophers have argued that not all criteria of identity conform to (CI). Timothy Williamson (1990), for instance, distinguishes criteria of identity of this kind – which he calls *one-level* criteria of identity – from criteria of identity of the following form:

$$\forall \zeta \forall \theta (\S(\zeta) = \S(\theta) \leftrightarrow \Phi(\zeta, \theta)), \qquad \text{(FCI)}$$

where § is an operator taking items of type ζ and θ to objects. As in the case of (CI), Φ must be an equivalence relation because identity is.

Williamson calls criteria of the form (FCI) *two-level* criteria, since they specify when an object $\S(\zeta)$ is identical to an object $\S(\theta)$ in terms of a relation between entities ζ and θ which are in principle distinct from $\S(\zeta)$ and $\S(\theta)$. Thus, whilst (CI) provides a criterion of identity for the objects falling under a certain concept in terms of a relation Φ holding between these very same objects, (FCI) provides

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a criterion of identity for objects of a certain kind in terms of a relation holding between entities which may belong to *another* kind.

Two-level criteria are sometimes called *Fregean*, since in *Die Grundlagen* and elsewhere Frege offered many examples of criteria of this kind. The first he considers has now become the paradigmatic example of a two-level criterion:

$$\forall x \forall y (D(x) = D(y) \leftrightarrow x \parallel y). \qquad \text{(Dir)}$$

(Dir) states that any two lines have the same direction if and only if they are parallel, and is a criterion of identity for *directions*. But the relation ∥ in terms of which (Dir) is formulated is a relation between *lines*, and this makes it clear that it is a two-level criterion. Within the neo-Fregean programme in the philosophy of mathematics (see, e.g., Wright 1983; Hale and Wright 2001), two-level criteria are also known as *abstraction principles*.

Philosophers disagree over the status of the relation between one- and two-level criteria. Williamson (1990; 1991), for instance, contends that neither type is reducible to the other, whilst E. J. Lowe (1989; 1991) claims that two-level criteria can be reduced to one-level criteria and, as a result, the latter are more fundamental. We do not need to enter this debate here, but take the opportunity to note that we will mostly be concerned with one-level criteria, although two-level criteria will also be discussed (see, in particular, Section 1.7 and Chapter 5).

Philosophers also disagree over what *kind of principles* criteria of identity are (see Horsten 2010 for an overview). On the face of it, identity criteria specify what it *is* for two things falling under a certain concept to be identical. As such, they appear to be *metaphysical* principles. However, it also seems to be the case that, just like criteria of application, criteria of identity partly determine the *meaning* of the term they are associated with. In this sense, one might regard them as *semantical* principles. Finally, it seems possible to use criteria of identity to *find out* whether two things falling under a certain concept are in fact one and the same: according to (Dir), I can find out whether the direction of line *a* is the same as *b*'s by finding out whether *a* and *b* are parallel. This aspect of identity criteria – no doubt responsible for their name – seems to make them *epistemic* principles.

The central disagreement, however, does not seem to concern so much whether identity criteria have the aforementioned roles, but rather which role is the *primary* one. Again, we do not need to adjudicate the matter here, since nothing we shall say hinges on the outcome of this dispute: all we shall assume is that identity criteria do have the roles in question.

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1.4 Extensionality

Is $[x : x \text{ is a set}]$ associated with a criterion of identity? It is typically thought that it is, and the reason offered is that we want to be able to count the members of a set: since sets can themselves be members of some sets,[8] it follows that we should be able to count sets themselves. And as Frege (1884: §54) observed, in order to count the objects falling under a concept C we need to be able to tell when any two such objects are in fact one and the same.

Frege's point is well taken: if we want to count the objects falling under a certain concept, it is crucial that we should count each object once, and only once. And that is why if we want to count sets, $[x : x \text{ is a set}]$ needs to be associated with a criterion of identity (see Dummett 1981: 546–549): such a criterion is required to avoid double-counting.

So when is it that two sets are identical? The standard view on the matter is that the identity conditions for sets are given by the Axiom of Extensionality, which asserts that if two sets have the same members, they are identical:

$$\forall x \forall y (\text{Set}(x) \wedge \text{Set}(y) \rightarrow (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y)). \qquad \text{(Ext)}$$

We will sometimes be concerned with theories which only deal with sets. In that case, the antecedent of (Ext) is unnecessary, and the Axiom of Extensionality takes the following form:

$$\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y). \qquad \text{(Ext}^\star\text{)}$$

Now since

$$\forall x \forall y (x = y \rightarrow \forall z (z \in x \leftrightarrow z \in y))$$

is a theorem of first-order logic with identity, Extensionality delivers the following extensional criterion of identity for sets:

$$\forall x \forall y (\text{Set}(x) \wedge \text{Set}(y) \rightarrow x = y \leftrightarrow (\forall z (z \in x \leftrightarrow z \in y))). \qquad \text{(ECS)}$$

We said earlier that criteria of identity are standardly taken to play various roles. One role is that of expressing, more or less directly, aspects of the nature of the objects falling under the concept with which they are associated. Another role is that of partly determining the meaning of the term they are associated with.

The extensional criterion of identity conforms to what we said on this score. For this criterion enables us to distinguish sets from other entities which in some sense collect a plurality into a unity and which do have a unique decomposition,

[8] If they weren't – and I hope I am allowed this counterpossible – then the strength of set theory would be so diminished as to compromise most of its use.

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