Finite Element Method for Solids and Structures

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A Concise Approach

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Preface

The finite element (FE) method is a powerful tool which improves the ability to engineer products or perform scientific studies through virtual prototyping and analysis. Inestimable costs associated with arduous synthesis/fabrication and testing have been saved by implementing FE analysis in engineering workflows. Today, most of the scientific and engineering workforce can access sophisticated FE software on their computers. These software tools offer capabilities backed by decades of research, enabling students and practicing engineers to easily develop and solve problems of high-level complexity. Software know-how is also not difficult to come by, as step-by-step tutorials on performing complex, expert-level modeling tasks can readily be found online.

With widespread adoption of commercial FE software, the needs of instructors and students of FE methods have also changed. The current trend is to increase student exposure to FE applications through software-based analyses. While this is both a welcome and a necessary approach, it comes at the cost of rigor and understanding the fundamentals of the FE method. At best, an early career can start with a working familiarity with the same tools used by more senior and experienced engineers. More common, however, is under-informed usage.

For most students of mechanical, aerospace, or civil engineering, a course on the FE method is often the first, if not the only, foray into computer-based mathematical modeling for engineering analysis. A solid grounding in FE theory can serve as a key launching point into other emerging computer-based modeling areas, including (at the time of writing) data science, machine learning, and artificial intelligence. The very same ideas of discrete data, linear algebra, vector computing, and computational algorithms in FE analysis are also foundational in these adjacent fields.

Accordingly, our textbook on the FE method can better prepare students for engineering practice without foregoing the essential mathematical foundations that could unlock broader career opportunities in such areas as database engineering writ large, quantitative analysis in the finance sector, and even computer graphics techniques used in movies and video games. This book can also be used by practicing engineers who want to learn more about the fundamentals of FE formulation. We feel that the best approach is to offer *a more refined and distilled perspective* of FE methods and yet be balanced with software tools, applications, and modern educational delivery platforms. We therefore planned this book with the following content:

– A text that offers a refined and distilled presentation of the FE method without compromising the theoretical foundations, but still remaining accessible to students with a basic college-level background in mathematics.

– A student's online guide comprised of (a) answers to some of the problems at the end of each chapter and (b) solutions to some of the example problems that benefit from the use of a programming language. The complete M-files will illustrate the calculations through programmable steps. This online material will be accessible via web icons in the margin of the text.
– An online instructor's supplement with additional MATLAB-based solutions and hand solutions to some of the problems at the end of each chapter, and higher-level MATLAB-based code examples. We emphasize that these materials are to provide additional resources to instructors. For instance, they may be used to (a) provide detailed solution steps that may be used by the instructor for exposition, (b) illustrate the basic implementation of FE methods in actual computer codes for advanced or graduate-level students, or (c) provide problems that can form the basis of special projects.
– A set of high-resolution, easy-to-follow walk-through videos presenting annotated examples of software applications, based on the ideas developed in the text. These use multiple modern software packages including PTC Creo Parametric, ANSYS APDL, ANSYS Workbench, and SolidWorks.

For practical reasons, we have tried to keep the online materials separate from the classical background and fundamentals in the text. History has shown that software and hardware can quickly grow obsolete, whereas the fundamentals can transcend the technologies with which we teach. By adopting this multi-pronged approach, we believe it is far easier to keep pace with evolving pedagogies while ensuring a consistent level of excellence from students. Web icons have been placed in the margins to identify where supplementary material is available online for students who would like more practice.

The authors have had the privilege of teaching courses on the FE method over many years at both undergraduate and graduate levels, while conducting research on the development of improved FE models. The present book is based on the course materials developed by the authors to help students understand the essentials of the FE method, primarily within the context of linear elasticity. Prerequisites for mathematics have been kept to the sophomore level, which includes calculus, elementary linear algebra, and ordinary differential equations. For some of the code-writing aspects, users of this book are expected to have a working knowledge of an interpreted or compiled programming language; MATLAB will be used primarily herein, but we employ a programming style that should make coding in other languages self-evident.

This book is organized into 11 chapters. Among these, the authors recommend that the first five chapters be used as materials for a one-semester course on the basics of the FE method for undergraduate students in mechanical, aerospace, and civil engineering. The instructor may also add the sections on one-dimensional (1D) heat transfer in Chapter 11 to their undergraduate course contents. After completing Chapter 1, the instructor may begin to use the video tutorials to introduce the FE analysis software and assign simple problems for practice throughout the duration of the course. The instructor may omit some of the topics to adjust to the pace and expectations of the students.

In Chapter 1, the FE formulation is introduced using a system of linear springs and a slender body undergoing uniaxial deformation. The body is divided into many segments or elements in which a linearly assumed displacement field is introduced. The concept of incremental work and strain energy is used to show how element stiffness matrices and load vectors are constructed and assembled into a global stiffness matrix and a global load vector to construct an FE equation. An example problem is used to demonstrate how well the solution obtained by the FE method compares with the exact solution.

In Chapter 2, truss structures are introduced within the context of an FE formulation in which a truss member is naturally an element. It shows how a global stiffness matrix is assembled from individual elements to determine displacements at the hinge joints and axial force in each member under both applied loads and temperature change. We then observe that torsional deformation of a slender body is mathematically equivalent to uniaxial or longitudinal deformation of the slender body.

In Chapter 3, we consider the FE formulation of slender bodies undergoing bending deformation. The Bernoulli–Euler beam bending theory is presented and exact solutions are obtained for example problems to be used as reference solutions. This is followed by the FE formulation of the two-node element for beam bending analysis. Finally, the two-node frame element is constructed combining the uniaxial element, torsional element, and bending elements.

Chapter 4 describes how the element mass matrix is constructed and assembled into a global mass matrix to set up the equation of motion, which can then be used for investigation of free vibration via eigenvalue analysis and forced vibration using numerical integration.

In Chapter 5, we consider bending of slender bodies under axial force. It is shown that the effect of axial force manifests as an effective stiffness matrix which can be used for static buckling analysis and free vibration analysis. It is then shown how the FE formulation can be used to investigate the dynamic stability of a slender body subjected to a compressive follower force.

Chapter 6 introduces the concept of virtual displacement and virtual work to express equilibrium in three-dimensional (3D) space in a scalar form to which the FE formulation can be applied. The equations described in Appendix 1 are used for this purpose.

Chapter 7 introduces mapping functions and shape functions used for the FE formulation. Mapping functions are used to map individual elements in the physical domain to the mapped domain for two-dimensional (2D) and 3D bodies. Shape functions identical to mapping functions are used for the assumed displacement in the isoparametric formulation. Integrations in the mapped domain are discussed, along with numerical integration.

Chapter 8 shows how the FE formulation is used to generate the element stiffness matrix and load vector for elements in the 2D domain. This is followed by the FE formulation in 3D space.

Chapter 9 describes FE modeling of thin plates and shells, beginning with the assumptions on kinematics of beam bending and plate bending and their effect on the strain–displacement and strain–stress relations. A salient feature of this chapter is the discussion on the solid element specifically tailored for modeling of plates and shells.

Chapter 10 describes element locking phenomena, in which elements lose the ability to deform when they are used to represent bending of slender and thin structures. It also shows that elements can exhibit locking when they are used to model incompressible solids. The effects of reduced-order integration are also discussed in conjunction with the spurious kinematic modes.

Chapter 11 describes the FE formulation of heat transfer in solids and structures, starting with steady-state heat conduction in the 1D domain. This is followed by the FE formulation of time-dependent heat transfer problems. The FE formulation is then extended to heat transfer problems in the 2D and 3D domains.

Appendix 1 provides the fundamental equations, such as the strain–displacement relation, force and moment equilibria, and constitutive equations to describe deformation of 3D solids and structures under applied loads. Appendix 2 describes Gaussian elimination with triple factorization for the solution of a system of linear equations and the skyline method for efficient storage of sparse matrices. There is a Bibliography supplied for further reading and supplementary information.

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